THESIS

Stationary Spacetimes and Inflation Universe from Intersecting M-branes

交差するＭブレインによる定常時空と指数膨張宇宙

Department of Physics, Waseda University

Makoto Tanabe

田辺誠

A doctoral dissertation submitted to
Department of Physics,

Waseda University

March, 2007

Utinam tam facile vera invenire possem quam falsa convincere.

CICERO

Contents

1 Introduction **7**
1.1 Black Holes in String Theory 8
1.1.1 Black Hole Thermodynamics 9
1.1.2 Extremal Reissner-Nordström Black Holes 12
1.1.3 Black Holes from D-brane 14
1.2 Cosmology in String Theory 15
1.2.1 NO-GO Theorem in String Theory 17
1.2.2 S-brane 18
1.3 Plan of the Thesis 20

2 Superstring and Supergravity **23**
2.1 Green-Schwarz Super String 23
2.1.1 Supercoset Parametrization 24
2.1.2 $\mathcal{G}$-invariant Action 25
2.2 Light-cone gauge quantization 30
2.2.1 GS Formalism in Light-cone Gauge 31
2.2.2 Hamiltonian Analysis 32
2.3 Superspace Formulation 35
2.3.1 Flat Superspace GS Formalism 35
2.3.2 Superspace Formulation of Supergravity 36

3 Black Brane Solutions **39**
3.1 Stationary Black Brane Solutions 40
3.1.1 Construction to Stationary Black Brane Solutions 41
3.1.2 M2 and M5-brane Solutions in M-theory 48
3.1.3 Supersymmetry in M2$\perp$M5 Black Branes 50
3.2 Null Dependent Brane Solutions 53
3.2.1 Construction to Null Brane Solutions 53
3.2.2 N2 and N5-brane solutions in M-theory 57
3.2.3 Supersymmetry in N2$\perp$N5 Solution 58

4 Black Hole Solutions **61**
4.1 Kaluza-Klein Compactification 62
4.2 Five-dimensional Black Hole Solutions 66
4.3 Hyperspherical Coordinates 69
4.3.1 BMPV Type Solutions: $\beta = 0$ 72
4.3.2 Brinkmann Wave Type Solutions: $\beta \neq 0$ 73
4.4 Hyperelliptical Coordinates 75
4.5 Hyperpolorical Coordinates 79

5 Cosmological Solution **85**
5.1 Inflation Universe from Intersecting N-branes 85

6 Concluding Remarks **89**

Chapter 1

Introduction

The aim of this thesis is to deepen our understanding of the Supergravity theories as a low-energy effective field theory of Super String and M-theory by black hole solutions and cosmology. Quantum Field Theory provides a successful way of describing the electromagnetic force as well as the strong and weak nuclear forces of the Standard Model. One of its major successes is that it describes these three fundamental forces by one single Quantum Field Theory with gauge group $SU(3) \times SU(2) \times U(1)$.

On the other hand, General Relativity provides a perfectly correct description of gravity as the fourth fundamental force. The main idea in General Relativity is that space and time are dynamical and are curved by the presence of matter and energy. the curvature accounts for the motion of objects in a gravitational field. However gravity is far weaker than electromagnetism and the nuclear forces, and this leads to difficulties to quantize General Relativity. In fact, it is not renormalisable, thus it fails to provide a description of gravity at the quantum level. Therefore, if there is to be a single Quantum Field Theory which unifies all four fundamental forces, then it must unify both General Relativity and the Standard Model in a consistent way.

Super String Theory is a one of the most possible way to unify the all forces, and only in ten-dimensional spacetime Super String has no anomalies for classical symmetry of the action. Therefore it can be renormalisable and we can obtain the quantum description in Super String in ten dimension. We can get the four-dimensional spacetime with the Standard Model gauge field theory by the process of compactification. This involves treating four of the ten dimensions as large and non-compact, to provide a four-dimensional spacetime, while regarding the remaining six dimensions as compact and too small to be detected with current experimentations. The requirement of Supersymmetry ensures that the number of bosonic and fermionic degrees of freedom are equal. It turns out that there are five distinct string theories as

- **Type I**: this consists of both open and unoriented closed strings;
- **Heterotic**: these are hybrid theories with closed strings and superstrings, and two distinct theories arise from considering the gauge group to be either $SO(32)$ or $E_8 \times E_8$;
- **Type IIA or Type IIB**: a theory of closed strings.

Theory	**Low-energy dynamics**
Type I String Theory	$N = 1$, $D = 10$ Supergravity / Yang-Milles with gauge group $SO(32)$
Type IIA String Theory	Non-chiral $N = 2$, $D = 10$ Supergravity
Type IIB String Theory	Chiral $N = 2$, $D = 10$ Supergravity
Heterotic String Theory	$N = 1$, $D = 10$ Supergravity / Yang-Milles with gauge group either $SO(32)$ or $E_8 \times E_8$
M-Theory	$N = 1$, $D = 11$ Supergravity

Table 1.1: Low energy effective field theory of Super String and M-theory.

There are five separate independent consistent theories, thus we wonder there is a single theory which unify the all of them. Indeed this was the cause of some concern, a lot of duality relationships were found between them.

Thus the five apparently different theories may be shown to be equivalent in a certain sense. Furthermore, it was found that these theories could actually all be unified by a single theory which requires eleven spacetime dimensions [1]. This unifying theory was named M-theory, and the different String Theories gives as perturbative expansions with respect to different limits.

To study the interactions of the massless fields in these Super String Theories, one must consider the low-energy effective actions. During the late seventies and early eighties, these actions were constructed for each of the five string theories. Remarkably, each was found to be described by a ten-dimensional Supergravity action, while the low-energy limit of M-theory can be described by Supergravity in eleven dimensions. The equations of motion for the bosonic fields can be derived from the beta-functions associated to the relevant sigma-model, when one imposes conformal invariance. The following table shows the correspondence between each theory and its associated low-energy effective limit is shown in Table 1.

So we see that theories of Supergravity are key points for the understanding of the dynamics of the massless fields which occur in Super String. This in itself is motivation enough to go ahead a greater understanding of Supergravity. In below we consider the two special case as Black Hole solutions and Cosmological solutions in Supergravity. We will have a deep understanding for the Super String and M-theory through them.

1.1 Black Holes in String Theory

Black holes are now one of the most important subjects in string theory. The Beckenstein-Hawking black hole entropy of an extreme black hole is obtained in string theory by statistical counting of the corresponding microscopic states [2]. However, it is very difficult to construct a black hole solution in string theory because of its strong coupling. So far, we know several interesting black hole solutions in supergravity theories [3, 4, 5, 6, 7, 8], which are obtained as an effective theory of a superstring model in a low energy limit. We also have black hole solutions in a higher-dimensional spacetime [9, 10], which play a key role in a unified theory such as string theory. In higher dimensions, there is no uniqueness theorem of black holes [11, 12, 13]. In fact, we have a variety of "black" objects such as a black brane [14, 15, 16, 17]. One of the most remarkable solutions is a black ring, which horizon has a topology of $S^1 \times S^2$ [18].

In this section, we introduce black holes in Super String theory, which is given by the bound state of D-branes. In paticular, the most interest case of black holes in string theory is BPS state. The BPS state is the balanced state of gravity and the other fields, and at this state the quantum state is equivalent to the classical solutions. In four dimension the BPS state is related to the extremal limit of black hole, thus we give some review about the properties of extremal Reissner-Nordström black hole.

1.1.1 Black Hole Thermodynamics

The Einstein-Maxwell action with Plank units ($c = G = \hbar = k_B = 1$) is

$$S = \int d^4x \, \frac{1}{16\pi} \left[\sqrt{-g}\mathcal{R} - \frac{1}{4} F_{\mu\nu}F^{\mu\nu} \right] , \tag{1.1}$$

where $\mathcal{R}$ is the Ricci scalar given by the metric $g_{\mu\nu}$, and $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$ is the field strength of an Abelian vector field A_μ.

The equations of motion of $g_{\mu\nu}$ and A_μ are written by

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu}\mathcal{R} = 8\pi T_{\mu\nu} \, , \tag{1.2}$$

$$\nabla_\mu F^{\mu\nu} = 0 \, , \tag{1.3}$$

where the energy-momentum tensor is

$$T_{\mu\nu} = F_{\mu\rho}F_\nu{}^\rho - \frac{1}{4} g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} \, . \tag{1.4}$$

The Bianchi identity $\nabla_\mu T^{\mu\nu} = 0$ is equivalent to the conservation law of the vector field as

$$\epsilon^{\mu\nu\rho}\nabla_\mu F_{\nu\rho} = 0 \, , \tag{1.5}$$

where $\epsilon^{\mu\nu\rho}$ is a completely antisymmetric tensor.

Assuming the static spherical metric,

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \, , \tag{1.6}$$

then we can solve the Maxwell's equations in vacuum as

$$F_{tr} = \frac{q}{r^2} \, , F_{\theta\phi} = p \sin\theta \, , \tag{1.7}$$

where q is the black hole's electric charge and p is the black hole's magnetic charge. The electric and magnetic charge can be written by the Gauss law,

$$q = \frac{1}{4\pi}\int_{S^2} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} dx^\mu dx^\nu = \frac{1}{4\pi}\int_{S^2} r^2 F_{tr} \sin\theta d\theta d\phi \,, \tag{1.8}$$

and the magnetic charge p is the same definition. The stress-energy tensor becomes a diagonal matrix, thus the Einstein equation can be solved under this solution as

$$f(r) = 1 - \frac{2M}{r} + \frac{p^2+q^2}{r^2} \,, \tag{1.9}$$

where M is total ADM mass of the black hole. This solution is called Reissner-Nordström black hole, and only this solution is allowed in the Einstein-Maxwell system.

The vector $\xi^\mu = \partial x^\mu/\partial t$ is a Killing vector of the Reissner-Nordström black hole. The Killing horizon is defined that the Killing vector becomes null on the horizon. The condition at the Killing horizon is that $\Phi \equiv -t^\mu t_\mu = -g_{tt} = f(r) = 0$, thus we find two horizon surface at

$$r_\pm = M \pm \sqrt{M^2 - (p^2+q^2)} \,, \tag{1.10}$$

where r_+ is called outer horizon and r_- is inner horizon. The ADM mass satisfied the condition $\sqrt{p^2+q^2} \leq M$ from the existence condition of the real solution. The degenerate horizon case ($r_+ = r_-$) is given by $\sqrt{p^2+q^2} = M$ which called extremal Reissner-Nordström black hole. The outer horizon is identified as the event horizon of the black hole and the inner horizon is an apparent horizon.

Considering the near event horizon limit ($r \sim r_+$), the metric function can be expanded by

$$f(r) \approx f'(r_+)(r - r_+) = 2\kappa(r - r_+) \tag{1.11}$$

where κ is named the surface gravity of the black hole which is given by

$$\kappa \equiv \frac{1}{2} f'(r_+) = \frac{r_+ - r_-}{2r_+^2} = \frac{\sqrt{M^2-(p^2+q^2)}}{r_+^2} \,. \tag{1.12}$$

In the extreme limit the surface gravity becomes $\kappa = 0$ and the Schwarzshild limit $(p = q = 0)$ it is $\kappa = 1/4M$. The killing vector ξ^μ is null on the event horizon, then we find $\Phi \equiv -\xi^\mu \xi_\mu = f(r) = 0$ on the horizon, and the normal vector of the horizon is given by

$$\nabla_\nu(-\xi^\mu \xi_\mu) = \Phi_{,\nu} = \frac{d\Phi}{dr}\partial_\nu r = 2\kappa \xi_\nu \,. \tag{1.13}$$

The Killing vector ν must be satisfied the Killing's equation

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0 \,, \tag{1.14}$$

and the second derivative of Killing vector is related to the Riemann tensor as

$$\nabla_\mu \nabla_\nu R^\sigma{}_{\nu\rho\mu}\xi_\sigma\,. \tag{1.15}$$

The Frobenius's theorem of the Killing vector on horizon ($\xi^\mu\xi_\mu = 0$) is given by

$$\epsilon^{\mu\nu\rho}\xi_\mu \nabla_\nu \xi_\rho = 0\,, \tag{1.16}$$

then using the Killing equation one can derive two equations

$$\kappa^2 = -\frac{1}{2}\nabla^\mu \xi^\nu \nabla_\mu \xi_\nu = \frac{1}{2}R_{\mu\nu}\xi^\mu\xi^\nu\,. \tag{1.17}$$

$$\xi^\mu \partial_\mu \kappa = 0\,. \tag{1.18}$$

The second one implies the surface gravity is constant along the Killing vector ξ^μ on the horizon. For the stationary black hole there is a time-like Killing vector, thus the surface gravity is constant for time.

The area of the event horizon of Reissner-Nordström black hole is

$$A = \int_{S^2} \sqrt{g_{\theta\theta}g_{\phi\phi}}\,d\theta d\phi = \int_{S^2} r^2 \sin\theta d\theta d\phi = 4\pi r_+^2 = 4\pi(M + \sqrt{M^2 - (p^2+q^2)})^2 \tag{1.19}$$

and Bekenstein and Hawking show that the entropy of the black hole is proportional to the area of the event horizon is defined as

$$S = \frac{A}{4}\,, \tag{1.20}$$

The variation of the entropy is given by

$$dS = \frac{1}{4}\left[\frac{\partial A}{\partial M}dM + \frac{\partial A}{\partial p}dp + \frac{\partial A}{\partial q}dq\right] = \frac{4\pi}{f'(r_+)}dM - \frac{4\pi p}{f'(r_+)r_+}dp - \frac{4\pi q}{f'(r_+)r_+}dq\,, \tag{1.21}$$

where we use the relation between the event horizon and the conservation quantities as

$$f'(r_+) = \frac{r_+ - r_-}{r_+^2} = \frac{\sqrt{M^2 - (p^2+q^2)}}{(M + \sqrt{M^2-(p^2+q^2)})^2}\,. \tag{1.22}$$

Therefore the derivation of the total mass can be identified as the first law of thermodynamics,

$$dM = \frac{f'(r_+)}{4\pi}dS + \frac{p}{r_+}dp + \frac{q}{r_+}dq = TdS + \chi dp + \phi dq\,, \tag{1.23}$$

where using the surface gravity $\kappa = 2f'(r_+)$, the Hawking temperature of the black hole is defined by

$$T = \frac{\kappa}{2\pi}\,, \tag{1.24}$$

and the magnetic and the electric potential on the horizon is defined by

$$\chi = \frac{p}{r_+}\,, \phi = \frac{q}{r_+}\,. \tag{1.25}$$

The zeroth law of thermodynamics is that the temperature is constant when the system is thermal equivalence, and in black hole thermodynamics the Hawking temperature is constant on the stationary black hole horizon. Hawking proved that the area of the event horizon will never decrease in any classical physical process, however the black hole is evaporating by the Hawking radiation, and the area surface is decreasing. Hawking consider the isolated system witch include the black hole and radiation, then the total entropy of the system is sum of the Bekenstein-Hawking entropy and the entropy of Hawking radiation. Thus the total entropy is increasing in any physical process. This result is equivalent to the second law of thermodynamics, which is that the total entropy of isolated system must no decrease ($dS_{\rm tot} \geq 0$). The third law of thermodynamics is that it is impossible for any process to reduce $S = 0$ in finite number of operations. Since the Bekenstein-Hawking entropy is proportional to the surface gravity, $S = 0$ is equivalent to the extremal Reissner-Nordström black hole. The third law in black hole is that it is impossible for any process to the extremal black hole in finite number of operations.

1.1.2 Extremal Reissner-Nordström Black Holes

The Hawking radiation is a big evidence of black hole thermodynamics, but there is a new problem named the information paradox. Black hole is a vacuum solution, thus a quantum state of black hole must be a pure state in natural. When black hole evaporates, the system of total entropy contains the Bekenstein-Hawking entropy and and the radiation entropy, thus the total system becomes a mixed state. Finally the black hole have vanished and the system contain only Hawing radiation. The spectrum of the Hawking radiation is thermal, thus the final state is a pure state again. In quantum mechanics the transition from pure state to mixed state is not allowed under the unitary transformation, thus there is a paradox between general relativity and quantum mechanics. To avoid the paradox, we must count up the micro state of black hole. In statistical mechanics the entropy is a logarithmic measure of a number of state as $S = -\log\Omega$. The analogy of this statement suggests that the black hole entropy can be described by the number of micro-states like

$$S = \log\Omega(M, p, q)\,, \tag{1.26}$$

where Ω can be determined by some quantum micro state.

To understand the quantum micro state, quantization of the gravity theory must be needed. However it is impossible for any approach expect string theory at this time. The low energy effect theory of string theory is named Supergravity, and the black hole solutions are the classical vacuum solutions in Supergravity. Black holes in Supergravity are also satisfied the black hole thermodynamics, and the entropy of the Supergravity is proportional to the are surface of black holes. Susskind [109] consider that a candidate for the origin of the entropy is fundamental string itself. Because of no hair theorem of black hole, the same mass and charge gives only one black hole solution in four dimension. The state of the same mass of the fundamental string is degenerated, and the degeneracy factor of the string with mass M is a number of state $\Omega \sim e^{\ell_s M}$, where ℓ_s is a string length. The ten dimensional Newton constant is related to the string length as $G_{10} \sim g^2 \ell_s^8$, thus the relation about four-dimensional Newton constant and string length is $G \sim g_s^2 \ell_s^2$. However we now use the Plank unit that gives $G = 1$, thus the string length is proportional to inverse of the coupling constant $\ell_s \sim g_s^{-1}$. The black hole entropy is derived from the area surface as

$$\Omega \sim e^{A/4} \sim e^{M^2}\,, \tag{1.27}$$

and if and only if the entropy of string and black hole is equivalent, than the string length becomes horizon radius $\ell_s \sim M$. At this time weak field approximation $g_s \ll 1$ is broken down, then the description by the black hole is better than by fundamental string.

This roughly argument is impossible to determined the coefficient of entropy because it use a strong coupling limit for transition from string to black hole. However it is well known that the BPS states in string theory will not change the degeneracy under the strong coupling limit. From the algebra of Supersymmetric generator of massive representation, the mass bound $M \geq Q$ is appeared, where Q is a central charge. The BPS state is determined by $M = Q$, and in the BPS state the degree of freedom are decreased by existence of the central charge. It is expected that the BPS state of fundamental string transit to the extreme Reissner-Nordström black hole. Because Reissner-Nordström black hole is also satisfied the mass bound

$$M \geq Q = \sqrt{p^2 + q^2}\,, \tag{1.28}$$

and in extremal limit it becomes $M = Q$. Therefore the extremal black hole entropy is equivalent to a logarithmic measure of a number of BPS state of string.

In extremal limit the metric function changes to

$$f(r) = \left(\frac{r - r_+}{r}\right)^2\,, \tag{1.29}$$

and the near horizon limit the metric becomes

$$ds^2 \approx -\frac{p^2 + q^2}{r^2}dt^2 + \frac{r^2}{p^2 + q^2}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\,, \tag{1.30}$$

which named the Bertotti-Robinson metric. Now we take the near horizon limit $r = r_+ + \epsilon\rho$ and $t = r_+^2 \tau/\epsilon$, then in the limit of $\epsilon \to 0$ the metric becomes

$$ds^2 = r_+^2 \left[\frac{-dt^2 + dz^2}{z^2} + d\theta^2 + \sin^2\theta d\phi^2\right]\,. \tag{1.31}$$

This metric represents the product of the two dimensional anti de Sitter (AdS) space times and a two-sphere with a constant radius r_+. The geometry of $AdS_2 \times S^2$ has a isometry $SO(1,2) \times SO(3)$, thus a test particle motion in the near horizon is realized as a one-dimensional conformal symmetries from $SO(1,2)$. Strominger showed that the near horizon AdS space is related to the conformal field theory (CFT) on AdS spcae, and the Bekenstein-Hawking entropy is equivalent to the entropy of CFT by the holographic description [21].

The extremal Reissner-Nordström black hole has the maximum value of the entopy, thus the entropy has the finite limit which is given by $S = \pi Q^2 = \pi(p^2 + q^2)$. The Hawking temperature of the extremal black hole is given by

$$T = \frac{\kappa}{2\pi} = \frac{1}{4\pi}\frac{r_+ - r_-}{r_+^2} \to 0\,, \tag{1.32}$$

thus there are no Hawking radiation with extremal black hole, and it is expected that the extremal black hole is a stable final state of black hole.

1.1.3 Black Holes from D-brane

In string theory there is another BPS state, which is D-branes. D-branes are the soliton solution in string theory, which expand p dimensional space named Dp-brane. D-branes have a charge

$$Q = \int d^{p+1}dx \epsilon^{\mu_1 \cdots \mu_{p+1}} A_{\mu_1 \cdots \mu_{p+1}}\,, \tag{1.33}$$

Strominger and Vafa expect that the BPS state of D-branes transit to the extremal black hole in strong coupling limit [2]. The Q number of Dp-brane mass is $M \sim Q(g_s \ell_s)^{-1}$ and the horizon radius can be estimate as $r_+ \sim GM \sim g_s Q$. In the strong coupling limit ($g_s \gg 1$), it must be need the quantization of gravity theory, but it is still difficult to calculate in string theory. In this below we take the strong coupling limit as $g_s Q \gg 1$, which means that for the large number of D-brane $Q \gg 1$ considering Supergravity, which is the low energy limit $g_s \ll 1$ of string theory, is allowed. In the strong coupling limit D-brane in the BPS state transit to the extremal black brane, which expand along the special dimension. Under the compactification along extra space direction,

The black brane solution is classical solution in Supergravity, but the brane is expanding along the space, thus in order to get a four dimensional black hole solution, compactification along the brane expanding direction is needed. However most of the black brane solution has a singularity on the horizon, which means that the expected value of the dilaton ϕ gives string coupling $g_s = e^\phi$ in four dimension, and the expect value of the dilaton diverges on the horizon. This result shows that the low energy limit $g_s \ll 1$ is no longer satisfied on horizon. Strominger showed that only a four charged black brane solution becomes regular horizon after compactification, thus black hole solution in four dimension is given by

$$ds^2 = -f(r)^{1/2}dt^2 + f^{-1/2}(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2)\,, \tag{1.34}$$

where metric function has four charges like

$$f(r) = \left(1 + \frac{Q_1}{r}\right)\left(1 + \frac{Q_2}{r}\right)\left(1 + \frac{Q_3}{r}\right)\left(1 + \frac{Q_4}{r}\right) . \tag{1.35}$$

The charges Q_i are depend on the D-brane configuration, e.g. and if we consider the special case which all the charges is equal $Q_1 = Q_2 = Q_3 = Q_4 = Q$, then under the coordinate transformation as $\tilde{r}^2 = r^2 + Q^2$, the metric becomes the extremal Reissner-Nordström black hole as

$$ds^2 = -\left(1 - \frac{Q}{\tilde{r}}\right)^2 dt^2 + \left(1 - \frac{Q}{\tilde{r}}\right)^{-2} d\tilde{r}^2 + \tilde{r}^2 d\theta^2 + \tilde{r}^2 \sin^2\theta d\phi^2 . \tag{1.36}$$

The horizon of the black hole is $\tilde{r} = Q$, thus horizon in ordinary metric (1.34) is given by $r = 0$. The Bekenstein-Hawking entropy is derived from the area surface of ordinary metric (1.34) with radius $r = 0$, which is

$$S = \frac{A}{4} = 2\pi\sqrt{Q_1 Q_2 Q_3 Q_4} . \tag{1.37}$$

Now we consider a example of brane configuration given by Maldacena [25]. Q_6 D6-branes are wrapped on $T^4 \times S_1 \times S_1'$, Q_2 D2-branes are wrapped on $S_1 \times S_1'$, Q_5 NS5-brane are wrapped on $T^4 \times S_1$, and wave along S_1 has a momentum N. This configuration represent a black brane solution in $\mathcal{N} = 8$ Supergravity. Because of the brane configuration the solution has $1/8$ BPS state, and after compactification along $T^4 \times S_1 \times S_1'$, the action in four dimensional spacetime provide $\mathcal{N} = 1$ Supersymmetry. The micro states of black brane is described by the state of open strings. One end of open string is exist on the D2-branes and the other end is exist on the D6-branes, and the open string carry the momentum in the background of NS5-branes. In the near horizon limit we can consider a CFT with effective central charge $c = 6Q_2Q_5Q_6$. The number of state in CFT with carrying N units of momentum is given by the Cardy formula as

$$\Omega(Q_2, Q_5, Q_6, N) \sim \exp\left(2\pi\sqrt{\frac{c_{\text{eff}}}{6}N}\right) \sim \exp\left(2\pi\sqrt{Q_2Q_5Q_6N}\right) . \tag{1.38}$$

This result is exactly the same in Bekenstein-Hawking entropy for the metric (1.34). In summary we can consider the micro states of black hole as a number of D-brane which is related to the degree of freedom of open strings. The black brane in Supergravity is still classical solution but it include the quantum description via open strings configuration. We will show later the black brane description of macroscopic and microscopic in generalizing with two rotating axis. In that case the representation of the entropy is changing a little.

1.2 Cosmology in String Theory

The Big Bang cosmology is based on certain evidences of some astronomical observation. First, the Hubble low explains the isotropic expansion of the universe, which is based on the observations of the redshift spectrum of distant galaxies and quasars. Next, the cosmic microwave background (CMB) proves a thermal equilibrium at the time of the recombination because of its homogeneous isotropic black body radiation spectrum. Then the COBE satellite observations showed that the universe is geometrically flat. The Big Bang nucleosynthesis predict that the mass ratio of helium and hydrogen is 1/4, and it was confirmed by the observation of hydrogen clouds of the galaxy.

Big Bang theory became the standard theory by the discovery of CMB, but some of the unresolved probrems were also left. One is a homogeneity isotropy problem, which is that the present homogeneous area of the universe is at least as large as the present horizon scale. Another is an initial flatness problem. The initial value of the density of matter and energy in the universe must be fine-turned, because of the flatness of the present universe . The other is an initial perturbation problem. The fluctuation of density of the universe has been suppressed by the expansion of the universe, thus the initial distribution of the matter is strongly inhomogeneous. These problems have been solved by the accelerated expansion in the early universe, named inflation.

The homogeneous isotropic universe is described by the FriedmannLematreRobertsonWalker (FLRW) metric as

$$ds^2 = -dt^2 + a(t)^2(dx^2 + dy^2 + dz^2)\,, \tag{1.39}$$

where the scale factor $a(t)$ is a function of time which represents the relative expansion of the universe. The Friedmann equation are derived from the Einstein equation

$$H^2 \equiv \left(\frac{1}{a}\frac{da}{dt}\right)^2 = \frac{8\pi G}{3}\rho\,. \tag{1.40}$$

where the Hubble constant is defined by $H = \dot{a}/a$, and the dotted sign represents the time differential. The energy density of the matter ρ follows the continuity equation which is derived from the Bianchi identity,

$$\dot{\rho} = -3H\left(\rho + P\right)\,, \tag{1.41}$$

where P is a pressure of the matter. For simplicity, we consider the de Sitter space by the vacuum energy, which is satisfied the equation of state $P = -\rho$. The continuity equation (1.41) can be solved as $\rho =$const. The Friedmann equation (1.40) can be also solved as an exponential expansion solution with the time of the expansion starting as t_i,

$$a(t) = a(t_i)e^{H(t-t_i)}\,. \tag{1.42}$$

Assuming that the expansion begins at the GUT vacuum energy scale $\rho \sim 10^{60}\,\mathrm{GeV}$, which derives the initial time is $t_i \sim 10^{-36}\,\sigma$. The ratio of the scale factor at the beginning and the time of the recombination can be estimate as $a(t_f)/a(t_i) = e^{100}$. Therefore the problems of Big Bang cosmology are resolved by the inflation.

1.2.1 NO-GO Theorem in String Theory

Theres are some no-go theorem, which the de Sitter universe is forbidden under the compactification from higher dimensional supergravity or superstring theory.

The first no-go theorem is showed by Gibbons [111]. Assume that D-dimensional Lorentzian manifold M is warped compactified by the compact n-dimensional Euclidean space Y, then four-dimensional Lorentzian manifold X is appeared. The metric can be described by the local coordinate x^M for M, x^μ for X and y^m for Y as

$$ds_D^2 = G_{MN}dx^M dx^N = w^2(y) g_{\mu\nu} dx^\mu dx^\nu + h_{mn} dy^m dy^n \,, \tag{1.43}$$

where $w(y)$ is the warp factor. Assuming the pure supergravity action as

$$S = \frac{1}{16\pi G}\int d^D x \sqrt{-G}\mathcal{R} + \int d^D x \mathcal{L}_{\text{matter}} \,, \tag{1.44}$$

where $\mathcal{L}_{\text{matter}}$ is the Lagrangian density for arbitrary matter. Using the Einstein equation in M,

$$R_{MN} = 8\pi G \left(T_{MN} - \frac{1}{D-2} g_{MN} T^L{}_L \right) \,, \tag{1.45}$$

where the bosonic energy-momentum tensor T_{MN} is defined by the matter Lagrangian $\mathcal{L}_{\text{matter}}$. All of the bosonic energy-momentum tensor in the pure gravity theory satisfies the strong energy condition

$$R_{MN} V^M V^N = 8\pi G \left(T_{MN} - \frac{1}{D-2} g_{MN} T^L{}_L \right) V^M V^N \geq 0 \,, \tag{1.46}$$

where V^M is a non-space-like vector. This condition means that local gravity is attractive. Now we assume extra but natural assumptions that Y is compact without boundary and the wrap factor $w(y)$ is smooth and nowhere vanishing, then the strong energy condition in M can be replaced to the condition in X as

$$R_{MN} V^M V^N = R_{\mu\nu} V^\mu V^\nu \geq 0 \,. \tag{1.47}$$

Now we assume X is a Einstein manifold, which has a constant curvature λ, then the condition becomes

$$R_{\mu\nu} V^\mu V^\nu = \lambda g_{\mu\nu} V^\mu V^\nu \geq 0 \,. \tag{1.48}$$

The arbitrary vector V^μ is non space-like, then we find $\lambda \leq 0$. Therefore the de Sitter space ($\lambda > 0$) is exclude.

Maldacena and Nunez [95] are generalized the Gibbons no-go theorem. They consider the action with non higher curvature correction and with the positive kinetic term of massless scalar field, which has a non-positive potential. They also assume that four-dimensional effective

Newton constant is finite, and the singularity witch is derived from vanishing wrap factor is only allowed. Under these conditions they showed the generalized no-go theorem, which means that de Sitter vacua is not allowed with such a compactification. However, there are several ways to avoid the no-go theorem.

One of the ways that non-compact internal manifold makes a de Sitter universe [112], but in this case four-dimensional Newton constant becomes zero, because the relation between the higher dimensional Newton constant and four-dimensional Newton constant is given by $G = G_D/V_n$ where V_n is the volume of the internal manifold. Considering the super-matters, which violate the strong energy condition [113] is possible to get the de Sitter solutions, but it is not clear to make such super-matters from compactification from string or M-theory. The most famous way to circumvent the no-go theorem is given by [38]. They added the α' correction to the leading order supergravity action, and also consider the localized non-trivial three-from fluxes. Furthermore they include nonperturbative quantum corrections from the fluxes, then they could find the de Sitter universe from type II string theory.

1.2.2 S-brane

As described the last section, a variety of ways to avoid the no-go theorem was appeared, but there are some kind of problem in each ways. In this thesis, I am interested in natural solutions embedding in string or M-theory, thus I consider the another way to overcome the no-go theorem. The idea is that the internal manifold can be depend on time. This first idea has been found by Townsend and Wohlfarth [66], and they found transient accelerating universe not the de Sitter space, which called S-brane means space-like brane. In this model our universe is parallel to the S-brane, thus considering S3-brane is the most simple, which relate to the four-form field strength F_4.

We consider the general higher-dimensional Einstein-Hilbert action with four-form flux

$$S = \frac{1}{16\pi G_D}\int d^Dx\sqrt{-g}\left(\mathcal{R}_D - \frac{1}{2\cdot 4!}F_4^2\right) . \tag{1.49}$$

Assuming the warped compactification from D-dimensional coordinate x^M in M to four dimensional coordinate $x{-}^{mu}$ in X via $n = D-4$ dimensional internal coordinate y^m in Y, then the metric in D-dimension is

$$ds^2 = G_{MN}dx^M dx^N = e^{-(D-4)\psi(x)}g_{\mu\nu}(x)dx^\mu dx^\nu + e^{2\psi(x)}h_{mn}dy^m dy^n , \tag{1.50}$$

where the internal compact manifold is the Einstein space with $R_{mn} = \sigma(n-1)h_{mn}$ and the field strength is satisfied the Hodge dual condition $*F_4 = b\,\mathrm{vol}(Y)$ with flux b. The inner manifold with $\sigma = -1$ represents hyperbolic compactification, $\sigma = 0$ is flat compactification, and $\sigma = +1$ is compactification by sphere.

Emparan and Garriga found the cosmological solution as [114]

$$ds_D^2 = -e^{6B+2n\psi}dt^2 + e^{2B}dx_3^2 + e^{2\psi}h_{mn}dy^m dy^n , \tag{1.51}$$

where dx_3^2 is a three-dimensional flat space and the metric functions are given by

$$B(t) = -\frac{1}{3}\log\left[b\sqrt{\frac{n-1}{6(n+2)}}\cosh 3(t-t_0)\right], \tag{1.52}$$

$$\psi(t) = \gamma(t) - \frac{3}{n-1}B(t), \tag{1.53}$$

where t_0 is an integral constant. The $\gamma(t)$ is depend on the curvature of the inner manifold as

$$\gamma(t) = \begin{cases} \frac{1}{n-1}\log\left[\beta\mathrm{csch}((n-1)\beta|t|)\right] & (\sigma=-1) \\ \beta t & (\sigma=0) \\ \frac{1}{n-1}\log\left[\beta\mathrm{sech}((n-1)\beta t)\right] & (\sigma=1) \end{cases}, \tag{1.54}$$

and β is defined by

$$\beta = \frac{1}{n-1}\sqrt{\frac{3(n+2)}{n}}. \tag{1.55}$$

In particular, the solution in string theory ($n=6$) represents an $S2-brane$, which are hypersurfaces with end point of the open strings, and the time coordinate obeys a Dirichlet boundary condition. Also in M-theory ($n=7$) represents an $SM2-brane$.

Under the compactification on inner manifold Y, the four-dimensional effective action appears that

$$S = \frac{1}{4\pi G_4}\int d^4x\sqrt{-g}\left[\mathcal{R}_4 - \frac{n(n+2)}{2}\left(\nabla\psi\right)^2 - V(\psi)\right], \tag{1.56}$$

where the metric function becomes a radion ψ on four-dimensional space, and the potential of the radion is given by

$$V(\psi) = \frac{b^2}{2}e^{-3n\psi} - \sigma n(n-1)e^{-(n+2)\psi}. \tag{1.57}$$

The radion fields ψ starts out at $\psi \to +\infty$ with large kinetic energy, and at the time of turning point ($d\psi/dt = 0, \psi = C(t)$) the field ψ, the energy of the field becomes potential dominant, then the accelerating universe has appeared. At the turning point the metric in M is

$$ds_D^2 = e^{6B+2nC}dt^2 + e^{2B}dx_3^2 + e^{2C}d\Sigma_{\sigma,n}^2, \tag{1.58}$$

and the four-dimensional Einstein metric becomes

$$ds_4^2 = g_{\mu\nu}dx^\mu dx^\nu = -e^{3(2B+nC)}dt^2 + e^{2B+nC}dx_3^2. \tag{1.59}$$

We can consider the metric is a flat FLRW universe with scale factor

$$a(t) = e^{B+nC/2}. \tag{1.60}$$

The four-dimensional proper time is obtained from

$$d\tau = a(t)^3 dt\,, \tag{1.61}$$

and the condition of the accelerating universe is determined by $d^2a/d\tau^2 > 0$. This accelerating condition is roughly equivalent to the positive potential energy at the turning point of the radion ψ. For the solution without any other extra fields, the case of no flux $(b = 0)$ requires the hyperbolic compactification $(\sigma < 0)$. The case of flux compactification $(b < 0)$ dose not have such a constraint, therefore the spherical compactification $(\sigma > 0)$ is also allowed. AT the big bang singularity the proper time becomes $\tau \to 0$, then the time coordinate starts from $t \to \infty$. In the hyperbolic case $(\sigma < 0)$, there is a singularity at $t = 0$, which gives $\tau \to \infty$, and the others case the time coordinate spans $-\infty \leq t \leq +\infty$.

All the case, the scale factor accelerate instantaneously only at the turning point. The standard inflationary cosmology in early universe requires a long period of acceleration, but the S-brane solutions only allow the instant acceleration. The S-brane model is also not fit the late time acceleration like a dark energy, because in this model the energy of the universe is kinetic dominant before the acceleration, but our universe have never been in kinetic dominant era in big bang scenario. There are so many extensions to avoid the short time acceleration, but in this thesis I consider the extension about null dependence not time dependence. Reminding the no-go theorem by Gibbons, the non spece-like arbitrary vector V^μ are considered, thus we can assume V^μ as a null vector. In this assumption the S-brane is also allowed the null coordinate dependence, thus it appropriate to call N-brane no longer.

1.3 Plan of the Thesis

This thesis is divided into two parts. First is the stationary spacetime formulation via intersecting M-branes, which related to the low dimensional black holes. The second part is the inflationary cosmological solutions from intersecting M-branes with Supersymmetry. In Chapter 2 we review Supergravity which is the low-energy effective field theory of superstring or M-theory. This is the our toolkit to solve the solutions we consider below. In Chapter 3 we solve the equation of motion with covariant constant spinor, which means that we have only solve the bosonic part of equations of motion. Especially we consider the extended solitonic object which is coupling with the anti-symmetric gauge field, named D-brane or M-brane. In this sense we will find the all metric components and the field can be described a harmonic functions, and these solitonic object must be satisfied the crossing rule. In Chapter 4 we discuss the compactification of the stationary intersecting black brane solutions from eleven dimension to five dimension via Kaluza-Klein compactification. This gives lower dimensional black hole solutions we want to consider, and under some symmetry of the base space, we will show various kind of black hole like object can be constructed. In Chapter 5 we analysis the intersecting null brane solutions, and after the Kalza-Klein compactification we will find five dimensional brane world like solutions. At that time we will get the four-dimensinal inflationary universe with collapsing extra bulk space solutions from null branes. In Chapter 6 we have conclusion and discussion.

Chapter 2

Superstring and Supergravity

On many occasions during the discussion of superstring theory we have obtained results that are consistent with presence of spacetime supersymmetry. These include the counting of the unmber of bosonic versus fermionic states at all mass levels, as well as the study of the spectrum at zero mass. the Super String. For String Theory, two-dimensional worldsheet Supersymmetry is easy to implement but what we want to realize is the spaceitme Supersymmetry in ten dimensions. There are two well-known formalisms for Supersymmetry is RNS formalism [39, 40] and GS formalism [41].

RNS formalism is based on Super-worldsheet and bosonic target manifold, thus Lorentz covariance is manifest, and Conformal Field Theory fit very well. We can also use BRST formalism for RNS string in convenient and powerful. However spacetime Supersymmetry emerges only after GSO projection, with infinite redundancy. The spacetime spinor are normally described by spin fields, but Ramond-Ramond bispinor fields are difficult to describe it.

On the other hand, GS formalism is based on bosonic worldsheet and target Supermanifold. To match up the worldsheet bosonic and fermionic degree of freedoms, we need local fermionic κ symmetry crucially. Classically it remain Lorentsz invariant, but first class and second class constraints are mixed in Lorentz-noninvariant way. Therefore we have some difficulty to quantization except in light-cone gauge.

In the conventional formalisms, not all the symmetries are manifestly realized quantum mechanically, but GS formulation is more promissing than RNS formalism. Thus we will show GS formalism in below.

2.1 Green-Schwarz Super String

In this section we introduce and study the GS formulation of superstring theory, in which spacetime supersymmetry is manifestly realized. Finally, we exhibit the low energy field theory approximations to the various superstring theories we have introduced, and identify them with known supergravity theories, in which we have local or gauged supersymmetry. We show how the GS formulation of superstrings may be used to couple strings to supergravity in a covariant way.

2.1.1 Supercoset Parametrization

A powerful way to construct the GS formalism in some important supergravity backgrounds is to employ the supercoset construction by Henneaux and Mezincescu [42], whcih regards the superstring as protagating in a coset supermanifold $\mathcal{K}$

$$\mathcal{K} = \mathcal{G}/\mathcal{H}\,, \tag{2.1}$$

where $\mathcal{G}$ is a non-compact supergroup and $\mathcal{H}$ is a bosonic subgroup of $\mathcal{G}$. The most important example of this formalism is flat $9+1$ dimensional spacetime. In this case $\mathcal{G}$ is $\mathcal{N}=2$ super-Poincaré group, which include $T \times SO(9,1)$ subgroup, and $\mathcal{H}$ is a Lorentz group $SO(9,1)$. Another interesting example is $AdS_5 \times S^5$ case, which is related to AdS/CFT correspondence given by Maldacena [43]. In this case $\mathcal{G}$ is given by $PSU(2,2|4) \supset SO(4,2) \times SO(6)$, and $\mathcal{H}$ is $SO(4,1) \times SO(5)$. It would be nice if one can treat the above two cases completely in parallel, at least classically.

The degree of freedom of GS super string in D-dimension is bosonic coordinate x^μ and fermionic coordinate $\theta^{\alpha I}$, where the indices are $\mu = 0, 1, \ldots, D-1$, $I = 1, 2, \ldots, \mathcal{N}$ and α is a spin index. $\mathcal{N}$ is a degree of extension of supersymmetric algebra of Poincare groups, i.e., type II superstring and M-theory give $\mathcal{N}=2$ and type I and heterotic string gives $\mathcal{N}=1$. $\theta^{\alpha I}$ is a Grassmann valued in D-dimensional spacetime and a scalar on worldsheet.

$\mathcal{N}=2$ GS string in flat backgournd gives the Gamma matrix Γ^μ which has 32 real symmetric components. The definition of $Gamma^\mu$ is

$$\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu}\,, \tag{2.2}$$

where $\eta_{\mu\nu} = (-,+,+,\ldots,+)$ is a Minkowski metric in 10 dimesional spacetime. We also define the new matrix

$$\Gamma_{10} = \Gamma^0\Gamma^1\cdots\Gamma^9 = \begin{pmatrix} \mathbf{1}_{16} & 0 \\ 0 & -\mathbf{1}_{16} \end{pmatrix}, \tag{2.3}$$

which gives chirality for the Majonara spinor and satisfied $\Gamma_10^2 = 1$.

Therefore we can choose the 16 componet of chiral and anti-chiral description as

$$\Lambda = \begin{pmatrix} \lambda^\alpha \\ \lambda_\alpha \end{pmatrix}, \quad \Gamma^\mu = \begin{pmatrix} 0 & (\gamma^\mu)^{\alpha\beta} \\ (\gamma^\mu)_{\alpha\beta} & 0 \end{pmatrix}, \tag{2.4}$$

where γ^μ satisfy the relations

$$2\eta^{\mu\nu}\delta^\gamma_\alpha = (\gamma^\mu)_{\alpha\beta}(\gamma^\nu)^{\beta\gamma} + (\gamma^\nu)_{\alpha\beta}(\gamma^\mu)^{\beta\gamma}\,, \tag{2.5}$$

$$(\gamma^{\mu\nu})_\alpha{}^\beta = \frac{1}{2}\left[(\gamma^\mu)_{\alpha\beta}(\gamma^\nu)^{\beta\gamma} - (\gamma^\nu)_{\alpha\beta}(\gamma^\mu)^{\beta\gamma}\right] = -(\gamma^{\mu\nu})^\alpha{}_\beta\,. \tag{2.6}$$

Generators of Super-Poincaré group $\mathbf{SP}_{N=2}$ are $\{L_{\mu\nu}, P_\mu, Q_{I\alpha}\}$ where $I = 1, 2$, $\mu = 0, 1, \ldots, 9$, and $\alpha = 1, 2, \ldots, 16$. They satisfy the algebra

$$[L_{\mu\nu}, P_\rho] = \eta_{\nu\rho} P_\mu - \eta_{\mu\rho} P_\nu \,, \tag{2.7}$$
$$[L_{\mu\nu}, Q_{I\alpha}] = \frac{1}{2} (\gamma_{\mu\nu})_\alpha{}^\beta Q_{I\beta} \,, \tag{2.8}$$
$$[P_\mu, P_\nu] = [P_\mu, Q_{I\alpha}] = 0 \,, \tag{2.9}$$
$$\{Q_{I\alpha}, Q_{J\beta}\} = -2i\delta_{IJ} (\gamma^\mu)_{\alpha\beta} P_\mu \,. \tag{2.10}$$

Decompose the superalgebra as $\mathbf{g} = \mathbf{h} + \mathbf{k}$, we find the algebra has the schematic structure

$$[\mathbf{h}, \mathbf{h}] = \mathbf{h} \,, \quad [\mathbf{h}, \mathbf{k}] = \mathbf{k} \,, \quad [\mathbf{k}, \mathbf{k}\} = \mathbf{k} \,. \tag{2.11}$$

Thus, $\mathcal{K}$ is not a symmetric space.

Now we parametrize the supercoset as

$$g(x, \theta) = e^{u(x,\theta)} \,, \quad u(x, \theta) = x^\mu P_\mu + \theta_I^\alpha Q_{I\alpha} \,, \tag{2.12}$$

where (x^μ, θ_I^α) forms the superspace in which the string propagates. s We will consider the type IIB case, where θ_I^α are both ten dimensional chiral. By using the Baker-Campbell-Hausdorff formula, we find

$$g(0, \epsilon) g(x, \theta) = g(x + i\epsilon_I \gamma^\mu \theta_I, \theta + \epsilon) \,, \tag{2.13}$$

thus we find the supersymmetric transformation as

$$\delta\theta^I = \epsilon^I \tag{2.14}$$
$$\delta\bar{\theta}^I = \bar{\epsilon}^I \tag{2.15}$$
$$\delta x^\mu = i\bar{\epsilon}^I \Gamma^\mu \theta^I \,, \tag{2.16}$$

where ϵ^I is a constant set of Majonara spinor.

2.1.2 $\mathcal{G}$-invariant Action

To construct the appropriate action, we need $\mathcal{G}$-invariant building blocks generated by the left-invariant Cartan 1-form current as

$$J = g^{-1} dg = J_i d\xi^i \,, \tag{2.17}$$

where ξ^i $(i = 0, 1)$ is the worldsheet coordinates. The J is the $\mathcal{G}$-invariant, because of under the global G transformation from the left which is given by $g \to Ug$. Thus J satisfies the fundamental Maurer-Cartan equation as

$$\begin{aligned} dJ &= dg^{-1} \wedge dg = -g^{-1} dg \wedge g^{-1} dg \\ &= -J \wedge J \,. \end{aligned} \tag{2.18}$$

Since the coset generators are closed as $[k, k\} = k$. J is valued entirely in k. Moreover, J can be computed explicitly as

$$J = L^A T_A = \Pi^\mu P_\mu + d\theta_I^\alpha Q_{I\alpha}\,, \tag{2.19}$$

where $\Pi^\mu \equiv dx^\mu - W^\mu$ is a invariant momentum 1-form and $W^\mu = i\sum_I \theta_I \gamma^\mu d\theta_I$. Maurer-Cartan equation (2.18) can reduce to

$$\begin{aligned} d\Pi^\mu &= -i\sum_I d\theta_I \gamma^\mu d\theta_I = -dW^\mu && (2.20)\\ d(d\theta_I^\alpha) &= 0\,. && (2.21)\end{aligned}$$

The action should be composed of the invariant 1-forms $L^A = (\Pi^\mu, d\theta_I^\alpha)$.

Define the inner product of forms of the same degree by

$$(\omega, \lambda) \equiv \int \omega \wedge *\lambda\,, \tag{2.22}$$

thus in two dimension we have

$$d\xi^i \wedge *d\xi^j = \sqrt{-g}g^{ij}d^2\xi\,, \tag{2.23}$$

where we define $\epsilon_{01} = -\epsilon^{01} = 1$. Now the kinetic term is easy to construct with the tension set to unit,

$$S_K = -\frac{1}{2}(\Pi^\mu, \Pi_\mu) = -\frac{1}{2}\int d^2\xi\sqrt{-g}g^{ij}\Pi_i^\mu \Pi_{\mu j}\,, \tag{2.24}$$

which is well known as the Polyakov action in string theory.

There are another important invariant one can construct, the Wess-Zumino term;

$$S_{WZ} = \int_{M_3} h\,, \tag{2.25}$$

where h is a invariant closed 3-forms, and M_3 is three-dimensional space such that ∂M_3 is the worldsheet we consider.

To construct h, we consider the mass dimensions of $\{H_i^\mu\}$ which is $\{-1, -1/2\}$, however h should have the same mass dimension as $\Pi_i^\mu \Pi_{\mu j}$, thus the only possibility of mass dimension -2 is

$$h = c_{IJ}\Pi^\mu(d\theta_I \gamma_\mu d\theta_J)\,, \tag{2.26}$$

where $c_{IJ} = c_{JI}$ is a symmetric tensor.

Since $d\theta_I$ are exact, using the Maurer-Cartan equation (2.20) we get

$$\begin{aligned} dh &= c_{IJ}d\Pi^\mu(d\theta_I\gamma_\mu d\theta_J)\\ &= -ic_{IJ}(d\theta_K\gamma^\mu d\theta_K)(d\theta_I\gamma_\mu d\theta_J)\,. \end{aligned} \tag{2.27}$$

Now using the Fierz identity;

$$(\gamma^\mu)_{\alpha\beta}(\gamma_\mu)_{\gamma\delta} + (\gamma^\mu)_{\beta\gamma}(\gamma_\mu)_{\alpha\delta} + (\gamma^\mu)_{\gamma\alpha}(\gamma_\mu)_{\beta\delta} = 0\,, \tag{2.28}$$

we easily get

$$dh = -i(c_{11} + c_{22})(d\theta_1\gamma^\mu d\theta_1)(d\theta_2\gamma_\mu d\theta_2)\,. \tag{2.29}$$

This is vanishing if and only if $c_{11} = -c_{22}$, and we redefine $c_{12} = c_{21} = 0$ by using a linear combination of θ_I. Thus we get h as

$$h = C_{WZ}\Pi^\mu d\tilde{W}_\mu\,, \tag{2.30}$$

where we define $\tilde{W}_\mu = W_{1\mu} - W_{2\mu}$, and $W_{I\mu} = i\theta_I\gamma_\mu d\theta_I$. Actually h is an exact form as

$$h = db = -d\left[C_{WZ}\left(\Pi^\mu\tilde{W}_\mu + \frac{1}{2}W^\mu\tilde{W}_\mu\right)\right]\,. \tag{2.31}$$

This can be easily checked b using the following identity, as well as the Maurer-Cartan equation (2.20) with the Fierz identity:

$$dW^\mu\tilde{W}_\mu = -W^\mu d\tilde{W}_\mu\,. \tag{2.32}$$

note that while h is composed entirely of invariant currents, but b is not currents.

First we introduce a general variation of the currents, which will be especially convenient for discussing the κ-symmetry later. Replace d in the definition of the current $J = g^{-1}dg$ by the variation δ and define

$$\bar{\delta}x \equiv g^{-1}\delta g = \bar{\delta}x^A T_A\,. \tag{2.33}$$

If we write $g = e^x$ with $x = x^A T_A$, $\bar{\delta}x$ is related to the usual variation δx by

$$\bar{\delta}x \sim \delta x - \frac{1}{2}[x, \delta x] + \cdots\,, \tag{2.34}$$

then it is straightforward to obtain

$$\delta J = d\bar{\delta}x + [J, \bar{\delta}x]\,. \tag{2.35}$$

This is of the same form as the infinitesimal right gauge transformation $J(gv)$ with $v = \exp(\bar{\delta}x)$ as

$$\begin{aligned} J(gv) &= (gv)^{-1}d(gv) = v^{-1}dv + v^{-1}Jv \\ &= J + d\bar{\delta}x + [J, \bar{\delta}x] + \mathcal{O}\left((\bar{\delta}x)^2\right)\,. \end{aligned} \tag{2.36}$$

For the flat space case, $\bar{\delta}x = \delta x^\mu - i\theta_I\gamma^\mu\delta\theta_I$ and $\bar{\theta}_I = \delta\theta_I$, thus the variation of the invariants are given by

$$\begin{aligned}
\delta J &= \delta\Pi^\mu P_\mu + \delta d\theta_I^\alpha Q_{I\alpha} \\
&= (d\bar{\delta}x^\mu)P_\mu + (d\bar{\delta}\theta_I^\alpha)Q_{I\alpha} + \left[\Pi^\mu P_\mu + d\theta_I^\alpha Q_{I\alpha}, \bar{\delta}x^\nu P_\nu + \bar{\delta}\theta_J^\beta Q_{J\beta}\right] \\
&= (d\bar{\delta}x^\mu + 2id\theta_I\gamma_\mu\bar{\delta}\theta_I)P_\mu + (d\bar{\delta}\theta_I^\alpha)Q_{I\alpha}\,. \qquad (2.37)
\end{aligned}$$

Therefore we find the variation of Π^μ and $d\theta_I^\alpha$ as

$$\begin{aligned}
&\delta\Pi^\mu = d\bar{\delta}x^\mu + 2id\theta_I\gamma_\mu\bar{\delta}\theta_I \\
&\delta(d\theta_I^\alpha) = d\bar{\delta}\theta_I^\alpha\,. \qquad (2.38)
\end{aligned}$$

We find the variation of S_K as

$$\begin{aligned}
\delta S_K &= -\int \delta\Pi^\mu \wedge *\Pi_\mu \\
&= -\int \left(d\bar{\delta}x^\mu + 2id\theta_I\gamma_\mu\bar{\delta}\theta_I\right) \wedge *\Pi_\mu \\
&= \int \left(\bar{\delta}x^\mu d * \Pi_\mu + 2i * \Pi_\mu d\theta_I\gamma_\mu\bar{\delta}\theta_I\right)\,, \qquad (2.39)
\end{aligned}$$

and another hidden contribution is from δg^{ij} as

$$\delta_g S_K = -\frac{1}{2}\int d^2\xi\sqrt{-g}\delta g^{ij}\left(\Pi_i^\mu\Pi_{\mu j} - \frac{1}{2}g_{ij}g^{k\ell}\Pi_k^\mu\Pi_{mu\ell}\right)\,. \qquad (2.40)$$

Now we can calculate the variation of S_{WS}, but first we consider the variation of the 3-form h, which is

$$\begin{aligned}
C_{WS}^{-1}\delta h &= \delta(\Pi^\mu d\tilde{W}_\mu) \\
&= \delta\Pi^\mu d\tilde{W}_\mu + i\Pi^\mu\delta(d\theta_1\gamma_\mu d\theta_1 - d\theta_2\gamma_\mu d\theta_2) \\
&= (d\bar{\delta}x^\mu + 2id\theta_I\gamma_\mu\bar{\delta}\theta_I)d\tilde{W}_\mu + 2i\Pi^\mu(d\theta_1\gamma_\mu d\bar{\delta}\theta_1 - d\theta_2\gamma_\mu d\bar{\delta}\theta_2) \\
&= d\left(\bar{\delta}x^\mu d\tilde{W}_\mu\right) + 2id\theta_I\gamma_\mu\bar{\delta}\theta_I d\tilde{W}_\mu - 2i\Pi^\mu d\left(d\theta_1\gamma_\mu\bar{\delta}\theta_1 - d\theta_2\gamma_\mu\bar{\delta}\theta_2\right) \\
&= d\left(\bar{\delta}x^\mu d\tilde{W}_\mu\right) d\left(2i\Pi^\mu\left(d\theta_1\gamma_\mu\bar{\delta}\theta_1 - d\theta_2\gamma_\mu\bar{\delta}\theta_2\right)\right)\,, \qquad (2.41)
\end{aligned}$$

where we use the $d\Pi^\mu = -dW^\mu$ and the Fierz identity. Therefore we find δh is exact, and the variation of the Wess-Zumino action can be written by

$$\delta S_{WZ} = C_{WS}\int\left(\bar{\delta}x^\mu d\tilde{W}_\mu + 2i\Pi^\mu\left(d\theta_1\gamma_\mu\bar{\delta}\theta_1 - d\theta_1\gamma_\mu\bar{\delta}\theta_1\right)\right)\,. \qquad (2.42)$$

To simplify we choose the $C_{WZ} = 1$, and the equations of motion are

$$d\left(*\Pi_\mu + \tilde{W}_\mu\right) = 0 \tag{2.43}$$
$$(*\Pi_\mu + \Pi_\mu)(\gamma^\mu d\theta_1)_\alpha = 0 \tag{2.44}$$
$$(*\Pi_\mu - \Pi_\mu)(\gamma^\mu d\theta_2)_\alpha = 0 \tag{2.45}$$
$$\Pi_i^\mu \Pi_{\mu j} - \frac{1}{2} g_{ij} g^{k\ell} \Pi_k^\mu \Pi_{\mu\ell} = 0\,. \tag{2.46}$$

We can get the relation between Π_μ and $*\Pi_\mu$, which is

$$*\Pi_\mu \pm \Pi_\mu = 2\sqrt{-g}\, d\xi^k \epsilon_{ki} P_\pm^{ij} \Pi_{\mu j}\,, \tag{2.47}$$

where $P_\pm^{ij}$ is the projection operators defined by

$$P_\pm^{ij} = \frac{1}{2}\left(g^{ij} \pm \epsilon^{ij}\sqrt{-g}\right) = -P_\mp^{ji}\,. \tag{2.48}$$

The projection operators of course satisfy the properties

$$P_+ + P_- = 1\,, \quad P_\pm P_\pm = P_\pm\,, \quad P_\pm P_\mp = 0\,, \tag{2.49}$$

and the index of i, j is changed by the metric g_{ij} as $P_{\pm j}^{i} = P_\pm^{ik} g_{kj}$. Especially in the conformal gauge $g_{ij} = \eta_{ij}$, we have $*d\xi^0 = d\xi^1$ and $*d\xi^1 = d\xi^0$, hence we find

$$*\Pi_\mu \pm \Pi_\mu = \left(\Pi_{\mu 0} \pm \Pi_{\mu 1}\right) d\xi^\pm\,, \tag{2.50}$$

where $\xi^\pm = \xi^0 \pm \xi^1$.

We introduce new type of variation named κ type variation, which is defined as the special case of the $\bar{\delta}$ variation as

$$\bar{\delta}_\kappa x^\mu = 0\,, \quad \bar{\delta}_\kappa \theta_I \neq 0\,. \tag{2.51}$$

From the previously developed formulas, we can get

$$\bar{\delta}_\kappa S = \bar{\delta}_\kappa^\theta S + \bar{\delta}_\kappa^g S = 0\,, \tag{2.52}$$

where

$$\begin{aligned}
\bar{\delta}_\kappa^\theta S &= 2i \int \left[(*\Pi_\mu + \Pi_\mu) d\theta_1 \gamma^\mu \bar{\delta}_\kappa \theta_1 + (*\Pi_\mu - \Pi_\mu) d\theta_2 \gamma^\mu \bar{\delta}_\kappa \theta_2\right] \\
&= 4i \int d^2\xi \sqrt{-g}\left(P_+^{ij} \partial_i \theta_1 \gamma_j \bar{\delta}_\kappa \theta_1 + P_-^{ij} \partial_i \theta_2 \gamma_j \bar{\delta}_\kappa \theta_2\right) &(2.53) \\
\bar{\delta}_\kappa^g S &= -\frac{1}{2} \int d^2\xi \bar{\delta}_\kappa (\sqrt{-g} g^{ij}) \Pi_i^\mu \Pi_{\mu j}\,. &(2.54)
\end{aligned}$$

We use the gamma matrix on the worldsheet $\gamma_i = \Pi_{\mu i}\gamma^\mu$, and $\bar{\delta}^\theta_\kappa S$ must be proportional to $\Pi^\mu_i \Pi_{\mu j}$, then $\bar{\delta}_\kappa \theta_I$ must be linear in $\Pi_{\mu i}$, and proportional to γ_i because of the Lorentz covariance. Thus the possible form is given by

$$\bar{\delta}_\kappa \theta_I = \gamma_i \kappa^i_I \,, \tag{2.55}$$

where κ^i_I is a local fermionic spinor parameter. Then we find

$$\bar{\delta}^\theta_\kappa = 4i \int d^2\xi \sqrt{-g} \left(P^{ij}_+ \partial_i \theta_i \gamma_j \gamma_k \kappa^k_1 + P^{ij}_- \partial_i \theta_2 \gamma_j \gamma_k \kappa^k_2 \right) \,, \tag{2.56}$$

and we can write down that

$$\begin{aligned} \gamma_j \gamma_k &= \frac{\{\gamma_j, \gamma_k\} + [\gamma_j, \gamma_k]}{2} \\ &= \Pi^\mu_j \Pi_{\mu j} - \frac{1}{2}\epsilon_{jk} \left(\epsilon^{\ell m} \gamma_\ell \gamma_m \right) \,. \end{aligned} \tag{2.57}$$

The second term is a obstacle for our case, thus we impose the projection conditions as

$$P^{ij}_+ \epsilon_{jk} \kappa^k_1 = 0 \,, \quad P^{ij}_- \epsilon_{jk} \kappa^k_2 = 0 \,, \tag{2.58}$$

and using the identity $P^{ij}_\pm \epsilon_{jk} = \pm\sqrt{-g} P^{ij}_\pm g_{jk}$, the projection condition becomes

$$P^{ij}_+ \kappa_{1j} = 0 \,, \quad P^{ij}_- \kappa_{2j} = 0 \,, \tag{2.59}$$

where $\kappa_I i = g_{ik} \kappa^k_I$. Under this projection condition, the variation becomes

$$\bar{\delta}^\theta_\kappa S = 4i \int d^2\xi \sqrt{-g} \left(P^{ij}_+ \partial_i \theta_1 \kappa^k_1 + P^{ij}_- \partial_i \theta_2 \kappa^k_2 \right) \Pi^\mu_j \Pi_{\mu k} \,, \tag{2.60}$$

thus κ-invariance holds

$$\bar{\delta}_\kappa (\sqrt{-g} g^{jk}) = 8i\sqrt{-g} \left(P^{ij}_+ \partial_i \theta_1 \kappa^k_1 + P^{ij}_- \partial_i \theta_2 \kappa^k_2 \right) \,. \tag{2.61}$$

Actually the right hand side of the Eq. (2.61) is symmetric and traceless in j, k as the left hand side.

2.2 Light-cone gauge quantization

So far the action is completely covariant but rather non-linear. The only known way to quantize this system is to gauge-fix the local symmetry. Now we fix the worldsheet general coordinate invariance by taking the conformal gauge $g_{ij} = \eta_{ij}$, which is note that the conformal invariance still remains at this stage. Next we fix the κ-symmetry by imposing the semi-light-cone gauge $\gamma^+ \theta_I = 0$, where $\gamma^\pm = \gamma^- \pm \gamma^9$. This is simplifies the action considerably because $\theta_I \gamma^+ d\theta_I = 0$ and $\theta_I \gamma^m d\theta_I = 0$ with $m = 1, \ldots, 8$, where we use the relation

$$-\frac{1}{4}(\gamma^+ \gamma^- + \gamma^- \gamma^+) = 1 \,. \tag{2.62}$$

2.2.1 GS Formalism in Light-cone Gauge

We define the light-cone components of a vector A^μ as $A^\pm = A^0 \pm A^9$, and then we can get the inner product as

$$A^\mu B_\mu = -\frac{1}{2}(A^+B^- + A^-B^+) + A^m B^m\,, \tag{2.63}$$

and we find the only non-zero components of W^μ and $\tilde{W}^\mu$ are W^- and $\tilde{W}^-$. In the other word we find $\Pi^+ = dx^+$, $\Pi^m = dx^m$ and $\Pi^- = dx^- - W^-$, and

$$\Pi^\mu \tilde{W}_\mu = -\frac{1}{2}\Pi^+\tilde{W}^- \; -\frac{1}{2}dx^+\tilde{W}^-\,, \tag{2.64}$$

and $W^\mu \tilde{W}_\mu = 0$, thus the action becomes simpler as

$$\begin{aligned} S_K &= -\frac{1}{2}\int d^2\xi \left[-\partial_i x^+ \left(\partial^i x^- - i\theta_I \gamma^- \partial^i \theta_I\right) + \partial_i x^m \partial^i x^m\right] &(2.65)\\ S_{WZ} &= \frac{1}{2}\int dx^+ \wedge \tilde{W}^- \\ &= -\frac{1}{2}\int d^2\xi \epsilon^{ij}\partial_i x^+ \left(i\theta_1\gamma^-\partial_j\theta_1 - i\theta_2\gamma^-\partial_j\theta_2\right)\,. &(2.66)\end{aligned}$$

Note that this action still contains the interaction of the form $x^+\theta\theta$. Finally we can choose the full light-cone gauge, which breaks the conformal invariance, $x^+ = q^+ + p^+\tau$, then the action becomes quadratic as

$$S = -\frac{1}{2}\int d^2\xi \left[\partial_i x^m \partial^i x^m - 2ip^+\left(\theta_1\gamma^-\partial_+\theta_1 + \theta_2\gamma^-\partial_-\theta_2\right)\right]\,, \tag{2.67}$$

where $\partial_\pm = (\partial_0 \pm \partial_1)/2$. Under the full light-cone gauge we find the free equations of motion as

$$\partial_+\partial_- x^m = 0\,, \quad \partial_+\theta_1 = \partial_-\theta_2 = 0\,. \tag{2.68}$$

We can write the GS action as

$$\begin{aligned} S &= \int d^2\xi \mathcal{L}_K + \mathcal{L}_{WZ} \\ &= -\frac{1}{2}\int d^2\xi \sqrt{-g}g^{ij}\Pi_i^m \Pi_{mj} + \int d^2\xi \epsilon^{ij}\left[\Pi_i^m\left(W_{mj} - \hat{W}_{mj}\right) - W_i^m \hat{W}_{mj}\right]\,, \end{aligned} \tag{2.69}$$

where the momenta is defined by

$$\Pi_i^m = \partial_i x^m - \sum_A W_i^{Am} = \partial_i x^m - \sum_A i\theta^A \gamma^m \partial_i \theta^A\,, \tag{2.70}$$

and $W_i^m = W_i^{1m}$ and $\hat{W}_i^m = W_i^{2m}$. Therefore we get the fermionic primary constraints as

$$D_\alpha^A = k_\alpha^A - i\left[k^m + \left(\partial_1 x^m - W_1^{1m}\right)(\gamma_m\theta^1)_\alpha + \left(\partial_1 x^m + W_1^{2m}\right)(\gamma_m\theta^2)_\alpha\right] = 0\,, \tag{2.71}$$

where $k^m = \partial\mathcal{L}/\partial\dot{x}_m$ and $k_\alpha^A = \partial\mathcal{L}/\partial\dot{\theta}_\alpha^A$.

2.2.2 Hamiltonian Analysis

To get the Hamiltonian we use the Arnowitt-Deser-Misner parametrization of the worldsheet metric as

$$ds^2 = -(Ndt)^2 + \gamma(d\sigma + N^1 dt)^2 \,. \tag{2.72}$$

The Hamiltonian density is given by

$$\mathcal{H} = N\sqrt{\gamma}T_0 + N^1 T_1 \,, \tag{2.73}$$

where

$$T_0 = \frac{1}{2}\left[\left(k - W_1 + \hat{W}_1\right)^2 + \Pi_1^2\right] \tag{2.74}$$

$$T_1 = \left(k - W_1 + \hat{W}_1\right) \cdot \Pi_1 \,, \tag{2.75}$$

with the notation $A^2 = A^m A_m$ and $A \cdot B = A^m B_m$.

To vanishing the momenta conjugate, we get the constraints $T_0 = T_1 = 0$, and more convenient combinations are

$$T_+ = \frac{T_0 + T_1}{2} = \frac{1}{4}\Pi^m \Pi_m \tag{2.76}$$

$$T_- = \frac{T_0 - T_1}{2} = \frac{1}{4}\hat{\Pi}^m \hat{\Pi}_m \,, \tag{2.77}$$

where Π^m and $\hat{\Pi}^m$ are given by

$$\begin{aligned} \Pi^m &= k^m - W_1^m + \hat{W}_1^m + \Pi_1^m \\ &= k^m + \partial_\sigma x^m - 2W_1^m \end{aligned} \tag{2.78}$$

$$\begin{aligned} \hat{\Pi}^m &= k^m - W_1^m + \hat{W}_1^m - \Pi_1^m \\ &= k^m - \partial_\sigma x^m + 2\hat{W}_1^m \,. \end{aligned} \tag{2.79}$$

We choose the conformal gauge $N/\sqrt{\gamma} = 1$ and $N^1 = 1$ in below, and the basic Poisson brackets is

$$\{x^m(\sigma), k^n(\sigma')\}_P = \eta^{mn}\delta(\sigma - \sigma') \tag{2.80}$$

$$\{\theta^{A\alpha}(\sigma), k_\beta^B(\sigma')\}_P = -\delta^{AB}\delta_\beta^\alpha \delta(\sigma - \sigma') \,, \tag{2.81}$$

and the rest of them is vanishing. In below we will concentrate on the left-moving sector without hat.

Constraint T_+ satisfies the Virasoro algebra of the form

$$\{T_+(\sigma), T_+(\sigma')\}_P = 2T_+(\sigma)\delta'(\sigma - \sigma') + \partial_\sigma T_+(\sigma)\delta(\sigma - \sigma') \,, \tag{2.82}$$

In fact another weakly vanishing quantity $t_+ = \partial_\sigma \theta^\alpha D_\alpha$, which commutes with T_+ forms the same Virasoro algebra. The total Virasoro generator is

$$T = T_+ + t_+ = \frac{1}{4}\Pi^m \Pi_m + \partial_\sigma \theta^\alpha D_\alpha \,, \tag{2.83}$$

and this generator satisfies also the Virasoro algebra as

$$\{T(\sigma), T(\sigma')\}_P = 2T(\sigma)\delta'(\sigma - \sigma') + \partial_\sigma T(\sigma)\delta(\sigma - \sigma') \,. \tag{2.84}$$

As for the fermionic constraint, we find

$$D_\alpha = k_\alpha - i(\gamma^m \theta)_\alpha (k_m + \partial_\sigma x_m) - (\gamma^m \theta)_\alpha (\theta \gamma_m \partial_\sigma \theta) \,, \tag{2.85}$$

and which satisfies that

$$\{D_\alpha(\sigma), D_\beta(\sigma')\}_P = 2i\gamma^m_{\alpha\beta}\Pi_m \delta(\sigma - \sigma') \,. \tag{2.86}$$

Since $\Pi^m \Pi_m$ is a constraint, we have the familiar situation that a half of D_α is of second class and the other half is of first class constraint.

The light-cone gauge is related to $SO(8)$ decomposition as

$$\gamma^+_{\dot{a}\dot{b}} = -2\delta_{\dot{a}\dot{b}} \,, \quad (\gamma^+)^{ab} = 2\delta^{ab} \tag{2.87}$$

$$\gamma^-_{ab} = -2\delta_{ab} \,, \quad (\gamma^-)^{\dot{a}\dot{b}} = 2\delta^{\dot{a}\dot{b}} \tag{2.88}$$

$$\gamma^i_{a\dot{b}}\gamma^i_{c\dot{d}} + \gamma^i_{a\dot{d}}\gamma^i_{c\dot{b}} = 2\delta_{ac}\delta_{\dot{b}\dot{d}} \,, \tag{2.89}$$

then we find the Poisson commutator as

$$\{D_a(\sigma), D_b(\sigma')\}_P = 2i\delta_{ab}\Pi^+ \delta(\sigma - \sigma') \,. \tag{2.90}$$

Instead of $D_{\dot{a}}$ we use the κ generator as

$$K_{\dot{a}} = D_{\dot{a}} - \frac{1}{\Pi^+}\Pi^i \gamma^i_{\dot{a}b} D_b \,, \tag{2.91}$$

and the bracket of $K_{\dot{a}}$ with D_b is given by

$$\{K_{\dot{a}}(\sigma), D_b(\sigma')\}_P = 4i\gamma^i_{\dot{a}c}\gamma^i_{b\dot{d}}\frac{1}{\Pi^+}\partial_\sigma \theta_{\dot{d}} D_c \delta(\sigma - \sigma') \,, \tag{2.92}$$

which is proportion to D_c. The brachet of $K_{\dot{a}}$ with itself is somewhat more involved and takes the form

$$\{K_{\dot{a}}(\sigma), K_{\dot{b}}(\sigma')\}_P = -8i\delta_{\dot{a}\dot{b}}\left(\mathcal{T} + \mathcal{K}\right)(\sigma)\delta(\sigma - \sigma') + D\mathrm{-term} \,, \tag{2.93}$$

where D-term signifies a term proportioanl to D_a and

$$\mathcal{T} = \frac{1}{\Pi^+} T\,, \quad \mathcal{K} = \frac{1}{\Pi^+} K_{\dot{c}} \partial_\sigma \theta_{\dot{c}}\,. \tag{2.94}$$

The new operator $\mathcal{T}$ and $\mathcal{K}$, which are proportional to the constraints, have the properties

$$\{\mathcal{T}(\sigma), \mathcal{T}(\sigma')\}_P = 0 \tag{2.95}$$

$$\{\mathcal{T}(\sigma), \mathcal{K}(\sigma')\}_P = \frac{\mathcal{K}}{\Pi^+}(\sigma)\delta'(\sigma - \sigma') \tag{2.96}$$

$$\{\mathcal{T}(\sigma), D_a(\sigma')\}_P = \frac{D_a}{\Pi^+}(\sigma)\delta'(\sigma - \sigma') \tag{2.97}$$

$$\{\mathcal{K}(\sigma), \mathcal{K}(\sigma')\}_P = -\frac{2\mathcal{K}}{\Pi^+}(\sigma)\delta'(\sigma - \sigma') - \partial\left(\frac{\mathcal{K}}{\Pi^+}\right)(\sigma)\delta(\sigma - \sigma') + D\text{−term} \tag{2.98}$$

$$\{\mathcal{K}(\sigma), D_a(\sigma')\}_P = -4i\frac{1}{H^{+2}}\gamma^i_{a\dot{b}}\gamma^i_{c\dot{d}} D_c \partial_\sigma \theta_{\dot{d}} \partial_\sigma \theta_{\dot{b}}(\sigma)\delta(\sigma - \sigma')\,. \tag{2.99}$$

We may now set $D_a = 0$ strongly by employing the Dirac bracket as

$$\begin{aligned}\{A(\sigma), B(\sigma')\}_D &= \{A(\sigma), B(\sigma')\}_P \\ &- \int d\sigma_1 d\sigma_2 \{A(\sigma), D_a(\sigma_1)\}_P C^{ab}(\sigma_1, \sigma_2)\{D_b(\sigma_2), B(\sigma')\}_P\,,\end{aligned} \tag{2.100}$$

where

$$C^{ab}(\sigma_1 \sigma_2) = \frac{1}{2i\Pi^+}\delta^{ab}\delta(\sigma_1 - \sigma_2)\,. \tag{2.101}$$

Under the Dirac bracket, θ_a becomes self-conjugate as

$$\{S_a(\sigma), S_b(\sigma')\}_D = i\delta_{ab}\delta(\sigma - \sigma')\,, \tag{2.102}$$

where $S_a = \sqrt{2\Pi^+}\theta_a$ is rescaled.

Now we adopt the semi light-cone gauge as $\gamma^+\theta = 0$, which means that $\theta_{\dot{a}} = 0$. Now we use the new notation $\phi_I = (\theta_{\dot{a}}, K_{\dot{a}})$, then the extended Dirac bracket is given by

$$\begin{aligned}\{A(\sigma), B(\sigma')\}_{D^*} &= \{A(\sigma), B(\sigma')\}_D \\ &- \int d\sigma_1 d\sigma_2 \{A(\sigma), \phi_I(\sigma_1)\}_D C^{IJ}(\sigma_1, \sigma_2)\{\phi_J(\sigma_2), B(\sigma')\}_D\,,\end{aligned} \tag{2.103}$$

where if $\phi_I = 0$ the C^{IJ} can be written by

$$C^{IJ}(\sigma_1, \sigma_2) = \delta\dot{a}\dot{b}\delta(\sigma - \sigma')\begin{pmatrix} 8i\mathcal{T} & -1 \\ -1 & 0 \end{pmatrix}\,. \tag{2.104}$$

We can set $\mathcal{K} = 0$ and we have only one constraint as

$$\{\mathcal{T}(\sigma), \mathcal{T}(\sigma')\}_{D^*} = 0\,, \tag{2.105}$$

and we can further adopt the full light-cone gauge by setting $x^+ = k^+ t$.

2.3 Superspace Formulation

In this section we will show the superspace formalism for Super String and its low energy effective field theory as Supergravity. Both of them can be described with ten-dimensional superspace coordinate.

2.3.1 Flat Superspace GS Formalism

We consider ten-dimensional N=2 superspace as $X^M = (x^m, \theta^{\mu I})$, where spacetime index $m = 0, 1, \ldots, 9$ and 32 Majonara spinor index $\mu = 1, \ldots, 16$, $I = 1, 2$ are gives $(10\|32)$ representation of superspace.

Now we define the flat superspace flame as

$$E^A = dX^M E_M^A \,, \tag{2.106}$$

where an index $A = (a, \alpha I)$ gives $SO(1,9)$ Poincare coordinate $a = 0, 1, \ldots, 9$ and $Spin(1,9)$ Majonara-Weyl spinor α. In the flat superspace $Spin(1,9)$ connection is vanishing. Thus a super derivative is defined as $\mathcal{D}_A = E_A^M \partial_M$, and the anti-commutation is

$$\{\mathcal{D}_{\alpha I}, \mathcal{D}_{\beta J}\} = 2(\Gamma^a)_{\alpha\gamma}(C^{-1})^{\gamma}_{\beta}\mathcal{D}_a \delta_{IJ} \,. \tag{2.107}$$

We can always pull back of spacetime frame E^A to the worldsheet, i.e.,

$$X^*(E^a) \equiv E_p^a d\xi^p = (\partial_p x^a i\bar{\theta}^I \Gamma^a \partial_p \theta^I) d\xi^p \tag{2.108}$$

$$X^*(E^{\alpha I}) \equiv E_p^{\alpha I} d\xi^p = \partial_p \theta^{\alpha I} d\xi^p \,, \tag{2.109}$$

thus we can put $\Pi_p^a = E_p^a$ and then we find

$$S_1[X] = \frac{1}{8\pi} \int_\Sigma d\mu_g g^{pq} E_p^a E_q^b \eta_{ab} \,. \tag{2.110}$$

Next we show S_2 is equivalent to the Wess-Zumino-Witten action, before that we put $\bar{E}^I \equiv (E^I)^T \Gamma^0$, where the index T means transpose of matrix. A three-form field is defined as

$$H \equiv E^a \wedge (\bar{E}^1 \Gamma_a \wedge E^1 \bar{E}^2 \Gamma_a \wedge E^2) \,, \tag{2.111}$$

and H must be satisfied the Bianchi identity as

$$\begin{aligned} dH &= dE^a \wedge (\bar{E}^1 \Gamma_a \wedge E^1 - \bar{E}^2 \Gamma_a \wedge E^2) \\ &= {}^i\bar{\theta}^I \Gamma^a d\theta^I \wedge (\bar{E}^1 \Gamma_a \wedge E^1 - \bar{E}^2 \Gamma_a \wedge E^2) \\ &= -i(\bar{E}^1 \Gamma^a E^1 + \bar{E}^2 \Gamma^a E^2) \wedge (\bar{E}^1 \Gamma_a \wedge E^1 - \bar{E}^2 \Gamma_a \wedge E^2) \\ &= -i(\bar{E}^1 \Gamma^a E^1 \bar{E}^1 \Gamma_a E^1 - \bar{E}^2 \Gamma^a E^2 \bar{E}^2 \Gamma_a E^2) \\ &= -(\Gamma^0 \Gamma^a)_{\alpha\beta} (\Gamma^0 \Gamma_a)_{\gamma\delta} E^{1\alpha} E^{1\beta} E^{1\gamma} E^{1\delta} - (1 \to 2) = 0 \,. \end{aligned} \tag{2.112}$$

The last terms E are symmetric for indices $(\alpha\beta\gamma\delta)$, but wedge products are anti-symmetric, thus dH is surely vanishing.

H can be written as

$$\begin{aligned}
H &= (dx^a - i\bar{\theta}^j\Gamma^a d\theta^j) \wedge (d\bar{\theta}^1\Gamma_a d\theta^1 - d\bar{\theta}^2\Gamma_a d\theta^2) \\
&= d(dx^a(\bar{\theta}^1\Gamma_a d\theta^1 - \bar{\theta}^2\Gamma_a d\theta^2)) \\
&\quad -i(\bar{\theta}^1\Gamma^a d\theta^1 d\bar{\theta}^1\Gamma_a d\theta^1 - \bar{\theta}^2\Gamma^a d\theta^2 d\bar{\theta}^2\Gamma_a d\theta^2) \\
&\quad -i(\bar{\theta}^2\Gamma^a d\theta^2 d\bar{\theta}^1\Gamma_a d\theta^1 - \bar{\theta}^1\Gamma^a d\theta^1 d\bar{\theta}^2\Gamma_a d\theta^2)\,,
\end{aligned} \tag{2.113}$$

where the second term is vanishing by Γ-identity. Therefore H becomes an exact form $H = db$, i.e.,

$$H = d\left[dx^a(\bar{\theta}^1\Gamma_a d\theta^1 - \bar{\theta}^2\Gamma_a d\theta^2) + i\bar{\theta}^1\Gamma^a d\theta^1\bar{\theta}^2\Gamma_a d\theta^2\right]\,, \tag{2.114}$$

and S_2 becomes Wess-Zumino-Witten action as

$$S_2[x] = -\frac{i}{4\pi}\int_B X^*(H)\,. \tag{2.115}$$

Using the worldsheet scalar $\kappa \equiv iE_p^a\Gamma_a(\kappa^{p1} + \kappa^{p2})$, we find supersymmetric transformation as

$$\delta X^M E_M^a \equiv \delta X^a = 0 \tag{2.116}$$
$$\delta X^M E_M^1 = (1+\Gamma)\kappa \tag{2.117}$$
$$\delta X^M E_M^2 = (1-\Gamma)\kappa\,, \tag{2.118}$$

thus we can define Γ-matrix as

$$\Gamma \equiv \frac{1}{2\sqrt{h}}\epsilon^{pq}E_p^a E_q^b \epsilon_{ab}\,, \tag{2.119}$$

where Γ-matrix satisfy $\Gamma^2 = 1$.

2.3.2 Superspace Formulation of Supergravity

We assume the diffeomorphism invariant $Diff(M)$ and local supersymmetry invariance, we get the group of diffeomorphism invariant of superspace.

First we consider the eleven-dimensional $\mathcal{N} = 1$ supergravity in superspace is given by local super coordinate $X^M = (x^m, \theta^\mu)$, where $m = 0, 1, \ldots, 10$ and $\mu = 1, \ldots, 32$. We can define a field E^A, Spin $(1, 10)$ connection and three-form as

$$E^A = dX^M E_M{}^A \tag{2.120}$$
$$\Omega_A{}^B = dX^M \Omega_{MA}{}^B \tag{2.121}$$
$$X = \frac{1}{3!}E^C \wedge E^B \wedge E^A X_{ABC}\,, \tag{2.122}$$

where the superspace index $A = (a, \alpha)$ relate the spacetime coordinate $a = 0, 1, \ldots, 10$ and $Spin(1,10)$ Majonara spin index $\alpha = 1, \ldots, 32$. The Spin connection satisfy that

$$\Omega_{ab} = -\Omega_{ba}\,, \quad \Omega_{\alpha b} = \Omega_{a\beta} = 0\,, \quad \Omega_{\alpha\beta} = \frac{1}{4}(\Gamma^{ab})_{\alpha\beta}\Omega_{ab}\,, \tag{2.123}$$

and three-form X is invariant under $Spin(1,10)$ transformation, which include three-from field $A_{\mu\nu\rho}$.

Using these forms we can find a curvature, a torsion and a field strength as

$$R_A{}^B \equiv d\Omega_A{}^B + \Omega_A{}^C \wedge \Omega_C{}^B \equiv \frac{1}{2}E^D \wedge E^C R_{CDA}{}^B \tag{2.124}$$

$$T^A \equiv dE^A + E^B \wedge \Omega_B{}^A \equiv \frac{1}{2}E^C \wedge E^B T_{BC}{}^A \tag{2.125}$$

$$H \equiv dX \equiv \frac{1}{4!}E^D \wedge E^C \wedge E^B \wedge E^A H_{ABCD}\,, \tag{2.126}$$

and also satisfy the Bianchi identity

$$DT^A = E^B \wedge R_B{}^A \tag{2.127}$$

$$DR_A^B = 0 \tag{2.128}$$

$$dH = 0\,, \tag{2.129}$$

where D is a covariant derivative with spin connection.

We find the field equations

$$\sum_{(ABC)} \left[R_{ABC}{}^D - D_A T_{BC}{}^D - T_{AB}{}^E T_{EC}{}^D\right] = 0 \tag{2.130}$$

$$\sum_{(ABC)} \left[D_A R_{BCD}{}^E + T_{AB}{}^F R_{FCD}{}^E\right] = 0 \tag{2.131}$$

$$\sum_{(ABCDE)} \left[D_A H_{BCDE} + T_{AB}{}^F H_{FCDE}\right] - \sum_{(ADBEC)} (-1)^{BC+BD+CE} T_{AD}{}^F H_{FBEC} = 0\,, \tag{2.132}$$

where (ABC) means cyclic transformation. However this field equation contains the unnecessary fields, thus we try to find the solutions witch satisfy the constraints. For the tangent space group structure (2.123), Eq. (2.131) is satisfied if Eq. (2.130) is. Thus we now solve the couplied Bianchi identities (2.130) and (2.132) simultaneoulsy.

Brink and Howe [44] find the torsion constraint as

$$T_{\alpha\beta}{}^\gamma = T_{\alpha b}{}^c = T_{ab}{}^c = 0 \tag{2.133}$$

$$T_{\alpha\beta}{}^c = -i(\Gamma^c)_{\alpha\beta} \tag{2.134}$$

$$H_{\alpha\beta\gamma\delta} = H_{\alpha\beta\gamma d} = T_{\alpha bcd} = 0 \tag{2.135}$$

$$H_{\alpha\beta cd} = i(\Gamma_{cd})_{\alpha\beta}\,. \tag{2.136}$$

Now we put the Riemann tensor $R_{ab} = \eta^{cd} R_{acbd}$ and the Ricci scalar $R = \eta^{ab} R_{ab}$, then the equation of motion with no fermionic fields $\theta = 0$ is

$$R_{ab} - \frac{1}{2}\eta_{ab} R = -\frac{1}{12} H_{acde} H_b^{cde} + \frac{1}{96}\eta_{ab} H_{cdef} H^{cdef} \tag{2.137}$$

$$D^a H_{abcd} = -\frac{1}{1728}\epsilon_{bcde_1\cdots c_4 f_1\cdots f_4} H^{e_1\cdots e_4} H^{f_1\cdots f_4}\,, \tag{2.138}$$

which is bosonic M-theoretical equation of motion.

Ten-dimensioanl $\mathcal{N} = 1$ supergravity in superspace is also in the same way as eleven-dimensional case we considered. A local supercoordinate is $X^M = (x^m, \theta^\mu)$ $(m = 0, 1, \ldots, 9,\ \mu = 1, \ldots, 16)$, and the field and spin $(10\|16)$ connection are

$$E^A = dX^M E_M^A \tag{2.139}$$

$$\Omega_A{}^B = dX^M \Omega_{MA}{}^B\,, \tag{2.140}$$

where $A = (a, \alpha)$ $(a = 0, 1, \ldots, 9,\ \alpha = 1, \ldots, 16)$, which is given by Spin $(1, 9)$ Majonara-Weyl spinor. In the Majonara-Weyl representation, the spin connection satisfy

$$\Omega_{ab} = -\Omega_{ba} \tag{2.141}$$

$$\Omega_{\alpha b} = \Omega_{a\beta} = 0 \tag{2.142}$$

$$\Omega_{\alpha\beta} = \frac{1}{4}(\Gamma^{ab})_{\alpha\beta}\Omega_{ab}\,. \tag{2.143}$$

The curvature two-form and torsion are determined by

$$R_A{}^B \equiv d\Omega_A{}^B + \Omega_A{}^C \wedge \Omega_C{}^B \equiv \frac{1}{2} E^D \wedge D^C R_{CDA}{}^B \tag{2.144}$$

$$T^A \equiv dE^A + E^B \wedge \Omega_B{}^A \equiv \frac{1}{2} E^C \wedge E^B T_{BC}{}^A\,, \tag{2.145}$$

and the Bianchi identity is

$$DT^A = E^B \wedge R_B{}^A \tag{2.146}$$

$$DR_A{}^B = 0\,. \tag{2.147}$$

Nilsson [45] and Witten [46] give torsion constraint as

$$T_{\alpha\beta}{}^\gamma = T_{\alpha b}{}^c = 0 \tag{2.148}$$

$$T_{\alpha\beta}{}^c = 2\Gamma_{\alpha\beta}{}^c \tag{2.149}$$

$$T_{a\beta}{}^\gamma = (\Gamma_a)_\beta \delta\phi^{\delta\gamma}\,, \tag{2.150}$$

where $\phi^{\delta\gamma}$ is exist. This conditions give the same equation we consider before.

Chapter 3

Black Brane Solutions

In this chapter we solve the brane solutions in Supergravity. In eleven dimension, we find the M-brane and in ten-dimesional type II Supergravity, we find the D-brane, F1-string and NS5-brane. Moreover, we show the intersecting multi-brane solutions which is related to the black hole and inflation universe. In below to simplify we only consider the bosonic action, which means that there is no back reaction from the fermion, and with the covariant constant fermion the bosonic action is closed by themselves. However we consider the condition of the covariant constant spinor which is related to the κ-symmetry and how exist the Supersymmetry in these solusions. We also ignore the Chern-Simons term in below, because under our null symmetric ansatz the Chern-Simons terms is vanishing automatically.

We now consider the effective field theory of Super String or M-theory. The action witch satisfy the equation of motion given by the last section, can be written by a general D-dimensional Supergravity bosonic action as

$$S = \frac{1}{16\pi G_D} \int d^D X \sqrt{-g} \left[\mathcal{R} - \frac{1}{2}(\nabla\varphi)^2 - \sum_A \frac{1}{2 \cdot n_A!} e^{a_A \varphi} F_{\mathbf{n_A}}^2 \right] , \tag{3.1}$$

where $\mathcal{R}$ is the Ricci scalar of a spacetime metric $g_{\mu\nu}$, $F_{\mathbf{n_A}}$ is the field strength of an arbitrary form with a degree $n_A (\leq D/2)$, and a_A is its coupling constant with a dilaton field φ. Each index A describes a different type of brane. Although we leave the spacetime dimension D free, the present action is most suitable for describing the bosonic part of $D = 10$ or $D = 11$ Supergravity.

The equations of motion are written in the following forms:

$$\begin{aligned} \mathcal{R}_{\mu\nu} &= \frac{1}{2}\partial_\mu\varphi\partial_\nu\varphi + \sum_A \Theta_{\mathbf{n_A}\mu\nu}, \\ \nabla^2\varphi &= \sum_A \frac{a_A}{2 \cdot n_A!} e^{a_A\varphi} F_{\mathbf{n_A}}^2, \\ \partial_{\mu_1}&(\sqrt{-g}\, e^{a_A\varphi} F_{\mathbf{n_A}}^{\mu_1\cdots\mu_{n_A}}) = 0 \,, \end{aligned} \tag{3.2}$$

where $\Theta_{\mathbf{n_A}\mu\nu}$ is the stress-energy tensor of the n_A-form, which is given by

$$\Theta_{\mathbf{n_A}\mu\nu} = \frac{1}{2 \cdot n_A!} e^{a_A \varphi} \left[n_A F_{\mathbf{n_A}\mu}{}^{\rho\cdots\sigma} F_{\mathbf{n_A}\nu\rho\cdots\sigma} - \frac{n_A - 1}{D-2} F^2_{\mathbf{n_A}} g_{\mu\nu} \right] . \tag{3.3}$$

We also have an additional equation, which is the Bianchi identity for the n_A-form, i.e.,

$$\partial_{[\mu} F_{\mathbf{n_A}\mu_1\cdots\mu_{n_A}]} = 0 \,. \tag{3.4}$$

This is automatically satisfied if we introduce the potentials of n_A-form.

3.1 Stationary Black Brane Solutions

As for a metric form for a spacetime with intersecting branes, we assume the following metric form [47]:

$$ds^2 = 2\theta^{\hat{u}}\theta^{\hat{v}} + \sum_{i=1}^{d-1} (\theta^{\hat{i}})^2 + \sum_{\alpha=2}^{p} (\theta^{\hat{\alpha}})^2 \,, \tag{3.5}$$

where $D = d + p$ and the dual basis $\theta^{\hat{a}}$ are given by

$$\theta^{\hat{u}} = e^{\xi} du \,, \quad \theta^{\hat{v}} = e^{\xi} \left(dv + f du + \frac{\mathcal{A}}{\sqrt{2}} \right) \,, \quad \theta^{\hat{i}} = e^{\eta} dx^i \,, \quad \theta^{\hat{\alpha}} = e^{\zeta_\alpha} dy^\alpha \,. \tag{3.6}$$

Here we have used light-cone coordinates; $u = -(t - y_1)/\sqrt{2}$ and $v = (t + y_1)/\sqrt{2}$. This metric form includes rotation of spacetime and a traveling wave. Since we are interested in a stationary solution, we assume that the metric components f, $\mathcal{A} = \mathcal{A}_i dx^i$, ξ, η and ζ_α depend only on the spatial coordinates x^i in d-dimensions, which coordinates are given by $\{t, x^i (i = 1, 2, \cdots, d-1)\}$. In this setting, we set each brane A in a submanifold of p-spatial dimensions, which coordinates are given by $\{y_\alpha (\alpha = 1, 2, \cdots, p)\}$. Note that the solution in this metric form is invariant under the gauge transformation, $\mathcal{A} \to \mathcal{A} + d\Lambda, \quad v \to v - \Lambda/\sqrt{2}$.

As for the n_A-form field with a q_A-brane, we assume that the source brane exists in the coordinates $\{y_1, y_{\alpha_2}, \cdots, y_{\alpha_{q_A}}\}$. The form field generated by an electric charge is given by the following form:

$$\begin{aligned} F_{\mathbf{n_A}} &= \partial_j E_A dx^j \wedge du \wedge dv \wedge dy_2 \wedge \cdots \wedge dy_{q_A} \\ &\quad + \frac{1}{\sqrt{2}} \partial_i B^A_j dx^i \wedge dx^j \wedge du \wedge dy_2 \wedge \cdots \wedge dy_{q_A} \,, \end{aligned} \tag{3.7}$$

where $n_A = q_A + 2$ and E_A and B^A_j are scalar and vector potentials. This setting automatically guarantees the Bianchi identity (3.4).

We can also discuss the form field generated by a magnetic charge by use of a dual $*n_A$-field with $*q_A$-brane, which is obtained by a dual transformation of the n_A-field with a q_A-brane

($*n_A \equiv D - n_A, *q_A \equiv *n_A - 2$). In other words, the field components of $F_{\mathbf{n_A}}$ generated by a magnetic charge are described by the same form of (3.7) of the dual field $*F_{\mathbf{n_A}} = F_{*\mathbf{n_A}}$. We then treat $F_{*\mathbf{n_A}}$, which is generated by a magnetic charge, as another independent form field with a different brane from $F_{\mathbf{n_A}}$, which is generated by an electric charge, when we sum up by the types of branes A.

3.1.1 Construction to Stationary Black Brane Solutions

To easy to calculate, we define the new variable as

$$\begin{aligned} H_A &= \exp\left[-\left(2\xi + \sum_{\alpha=\alpha_2}^{\alpha_{q_A}} \zeta_\alpha - \frac{1}{2}\epsilon_A a_A \varphi\right)\right] && (3.8)\\ V &= \exp\left[2\xi + (d-3)\eta + \sum_{\alpha=2}^{p} \zeta_\alpha\right], && (3.9)\end{aligned}$$

where

$$\epsilon_A = \begin{cases} +1 & n_A\text{-form field } (F_{\mathbf{n_A}}) \\ -1 & \text{the dual field } (*F_{\mathbf{n_A}}) \end{cases} . \tag{3.10}$$

Ten we find the basic equations as follows:

$$\partial^2 f + \partial_j f \partial^j \ln V = \frac{1}{8} e^{2(\xi-\eta)} \left[\mathcal{F}_{ij}^2 - \frac{1}{2}\sum_A \left(\mathcal{F}_{ij}^{(A)}\right)^2\right], \tag{3.11}$$

$$\partial^2 \xi + \partial_j \xi \partial^j \ln V = \frac{1}{2(D-2)} \sum_A (D - q_A - 3) H_A^2 (\partial E_A)^2, \tag{3.12}$$

$$\partial^j \mathcal{F}_{ij} + \mathcal{F}_{ij} \partial^j \left[2(\xi - \eta) + \ln V\right] = \sum_A H_A \mathcal{F}_{ij}^{(A)} \partial^j E_A, \tag{3.13}$$

$$\begin{aligned} &\left(\partial^2 \eta + \partial_l \eta \partial^l \ln V\right) \delta_i^j + 2\partial_i \xi \partial^j \xi + (d-3)\partial_i \eta \partial^j \eta + \sum_{\alpha=2}^{p} \partial_i \zeta_\alpha \partial^j \zeta_\alpha \\ &\quad + \partial_i \partial^j \ln V - \left(\partial_i \eta \partial^j \ln V + \partial^j \eta \partial_i \ln V\right) \\ &= -\frac{1}{2}\partial_i \varphi \partial^j \varphi + \frac{1}{2}\sum_A H_A^2 \left[\partial_i E_A \partial^j E_A - \frac{q_A + 1}{(D-2)} (\partial E_A)^2 \delta_i^j\right], \end{aligned} \tag{3.14}$$

$$\partial^2 \zeta_\alpha + \partial_j \zeta_\alpha \partial^j \ln V = \frac{1}{2(D-2)} \sum_A \delta_{\alpha A} H_A^2 (\partial E_A)^2, \tag{3.15}$$

$$\partial^2 \varphi + \partial_j \varphi \partial^j \ln V = -\frac{1}{2}\sum_A \epsilon_A a_A H_A^2 (\partial E_A)^2, \tag{3.16}$$

$$\partial_j (H_A^2 V \partial^j E_A) = 0, \tag{3.17}$$

$$\partial^j \left(V \mathcal{F}_{ij}^{(A)}\right) = 0\,, \tag{3.18}$$

where ∂_i is a partial derivative $\partial/\partial x^i$ in a flat $(d-1)$-space, $\partial^2 \equiv \partial_i \partial^i$, and $\mathcal{F}_{ij}$, $\mathcal{F}_{ij}^{(A)}$, and $\delta_{\alpha A}$ for each coordinate α are defined by

$$\begin{aligned}
&\mathcal{F}_{ij} \equiv \partial_i \mathcal{A}_j - \partial_j \mathcal{A}_i \\
&\mathcal{F}_{ij}^{(A)} \equiv 2H_A \left(\mathcal{A}_{[i}\partial_{j]}H_A - \partial_{[i}B^A_{j]}\right) \\
&\delta_{\alpha A} \equiv \begin{cases} D - q_A - 3 & \alpha = \alpha_2, \ldots, \alpha_{q_A} \\ -(q_A+1) & \text{otherwise} \end{cases} .
\end{aligned} \tag{3.19}$$

The square bracket denotes the anti-symmetrization of indices, i.e., $X_{[i}Y_{j]} \equiv \frac{1}{2}(X_iY_j - X_jY_i)$

Since E_A appears just with a spatial derivative ∂_i, we can replace it with $\tilde{E}_A = E_A - E_A^{(0)}$, where $E_A^{(0)}$ is a constant, which is fixed by a boundary condition. Using Eqs. (3.17) and (3.18), we obtain from Eqs. (3.12), (3.15), (3.16) and (3.13),

$$\partial^j \left[V \left(\partial_j \xi - \frac{1}{2(D-2)} \sum_A (D - q_A - 3) H_A^2 \tilde{E}_A \partial_j \tilde{E}_A \right) \right] = 0, \tag{3.20}$$

$$\partial^j \left[V \left(\partial_j \zeta_\alpha - \frac{1}{2(D-2)} \sum_A \delta_{\alpha A} H_A^2 \tilde{E}_A \partial_j \tilde{E}_A \right) \right] = 0, \tag{3.21}$$

$$\partial^j \left[V \left(\partial_j \varphi + \frac{1}{2} \sum_A \epsilon_A a_A H_A^2 \tilde{E}_A \partial_j \tilde{E}_A \right) \right] = 0\,. \tag{3.22}$$

This set of equations is a coupled system of elliptic-type differential equations, for which it is very difficult to find general solutions. Hence, in this paper, we assume the following special relations:

$$\partial_j \xi = \frac{1}{2(D-2)} \sum_A (D - q_A - 3) H_A^2 \tilde{E}_A \partial_j \tilde{E}_A, \tag{3.23}$$

$$\partial_j \zeta_\alpha = \frac{1}{2(D-2)} \sum_A \delta_{\alpha A} H_A^2 \tilde{E}_A \partial_j \tilde{E}_A, \tag{3.24}$$

$$\partial_j \varphi = -\frac{1}{2} \sum_A \epsilon_A a_A H_A^2 \tilde{E}_A \partial_j \tilde{E}_A\,, \tag{3.25}$$

which guarantee Eqs. (3.20), (3.21) and (3.22) to be correct.

These equations are relations between the first-order derivatives of variables just as the BPS conditions. The existence of Supersymmetry in the obtained solutions is shown in the next section. Hence, these relations may be related to a BPS state, or an extremal black brane solution in Supergravity. In fact, if $\mathcal{F}_{ij}^{(A)}$ is proportional to $\mathcal{F}_{ij}$ and $\mathcal{F}_{ij}$ is self-dual, we prove that 1/8 Supersymmetry remains in the solutions with M2⊥M5 branes for $D = 11$ Supergravity theory.

η is obtained from ξ, ζ_α and φ as

$$\begin{aligned}\partial^j\eta &= -\frac{1}{d-3}\partial^j\left(2\xi+\sum_{\alpha=2}^{p}\zeta_\alpha-\ln V\right)\\ &= -\frac{1}{2(D-2)}\sum_A(q_A+1)H_A^2\tilde{E}_A\partial^j\tilde{E}_A+\frac{1}{(d-3)}\partial^j\ln V\,. \end{aligned}\tag{3.26}$$

This gives

$$\begin{aligned}\partial^2\eta+\partial_j\eta\partial^j\ln V &= \frac{1}{V}\partial_j(V\partial^j\eta)\\ &= -\frac{1}{2V(D-2)}\sum_A(q_A+1)\partial_j(H_A^2V\tilde{E}_A\partial^j\tilde{E}_A)+\frac{1}{(d-3)V}\partial^2V\\ &= -\frac{1}{2(D-2)}\sum_A(q_A+1)H_A^2(\partial\tilde{E}_A)^2+\frac{1}{(d-3)V}\partial^2V\,. \end{aligned}\tag{3.27}$$

We have, however, another equation for η, i.e., Eq. (3.14), which should be satisfied as well. We have to find a solution which satisfies both equations. This consistency gives two conditions for $\tilde{E}_A$. In order to derive them, we first take a trace of Eq. (3.14), which leads to

$$\begin{aligned}&(d-1)(\partial^2\eta+\partial_l\eta\partial^l\ln V)+2(\partial\xi)^2+(d-3)(\partial\eta)^2+\sum_{\alpha=2}^{p}(\partial\zeta_\alpha)^2+\partial^2\ln V-2\partial_l\eta\partial^l\ln V\\ &=-\frac{1}{2}(\partial\varphi)^2+\frac{1}{2(D-2)}\sum_A[D-2-(q_A+1)(d-1)]H_A^2(\partial\tilde{E}_A)^2\,. \end{aligned}\tag{3.28}$$

Substituting Eqs. (3.23), (3.24), (3.25) and (3.27) into Eq. (3.28), we find the first condition:

$$\begin{aligned}&\frac{1}{2}\sum_{A,B}M_{AB}H_A^2H_B^2\tilde{E}_A\tilde{E}_B(\partial\tilde{E}_A)(\partial\tilde{E}_B)\\ &\quad-\sum_A H_A^2(\partial\tilde{E}_A)^2+4\left(\frac{d-2}{d-3}\right)V^{-1/2}\partial^2V^{1/2}=0\,, \end{aligned}\tag{3.29}$$

where

$$\begin{aligned}M_{AB} &= \frac{1}{(D-2)^2}\left[2(D-q_A-3)(D-q_B-3)\right.\\ &\quad\left.+(d-3)(q_A+1)(q_B+1)+\sum_{\alpha=2}^{p}\delta_{\alpha A}\delta_{\alpha B}\right]+\frac{1}{2}\epsilon_A\epsilon_B a_A a_B\,. \end{aligned}\tag{3.30}$$

We have also a traceless part of Eq. (3.14), which is written as

$$
\begin{aligned}
&2\left(\partial_i\xi\partial^j\xi-\frac{1}{d-1}(\partial\xi)^2\delta_i^j\right)+(d-3)\left(\partial_i\eta\partial^j\eta-\frac{1}{d-1}(\partial\eta)^2\delta_i^j\right)\\
&+\sum_{\alpha=2}^{p}\left(\partial_i\zeta_\alpha\partial^j\zeta_\alpha-\frac{1}{d-1}(\partial\zeta_\alpha)^2\delta_i^j\right)+\frac{1}{2}\left(\partial_i\varphi\partial^j\varphi-\frac{1}{d-1}(\partial\varphi)^2\delta_i^j\right)\\
&-\frac{1}{2}\sum_A H_A^2\left(\partial_i\tilde{E}_A\partial^j\tilde{E}_A-\frac{1}{d-1}(\partial\tilde{E}_A)^2\delta_i^j\right)+\partial_i\partial^j\ln V-\frac{1}{d-1}\partial^2\ln V\delta_i^j\\
&-\left(\partial_i\eta\partial^j\ln V+\partial^j\eta\partial_i\ln V-\frac{2}{d-1}(\partial\eta)(\partial\ln V)\delta_i^j\right)=0\,.
\end{aligned}
\tag{3.31}
$$

This equation gives the second condition:

$$
\begin{aligned}
&\frac{1}{2}\sum_{A,B}M_{AB}H_A^2H_B^2\tilde{E}_A\tilde{E}_B(\partial_i\tilde{E}_A)(\partial^j\tilde{E}_B)-\sum_A H_A^2\partial_i\tilde{E}_A\partial^j\tilde{E}_A\\
&-\frac{1}{d-1}\delta_i^j\left[\frac{1}{2}\sum_{A,B}M_{AB}H_A^2H_B^2\tilde{E}_A\tilde{E}_B(\partial\tilde{E}_A)(\partial\tilde{E}_B)-\sum_A H_A^2(\partial\tilde{E}_A)^2\right]\\
&-2(d-3)V^{\frac{1}{d-3}}\left[\partial_i\partial^j\left(V^{-\frac{1}{d-3}}\right)-\frac{1}{d-1}\delta_i^j\partial^2\left(V^{-\frac{1}{d-3}}\right)\right]=0\,,
\end{aligned}
\tag{3.32}
$$

We have to find a solution for two conditions (3.29) and (3.32). Here we shall assume V= constant. We shall also impose the condition $M_{AB}=0$ for $A\neq B$, which is called the intersection rule [48, 78, 50, 51]. This rule is derived in the case of spherically symmetric spacetime from the condition that each E_A is independent. Our case is just an ansatz.

Suppose that the q_A-brane and q_B-brane are filled in different spatial dimensions, but those branes are crossing on $\bar{q}_{AB}$ dimensions ($\bar{q}_{AB}<q_A,q_B$). Calculating (3.30), we obtain

$$
M_{AB}=\bar{q}_{AB}+1-\frac{(q_A+1)(q_B+1)}{D-2}+\frac{1}{2}\epsilon_A a_A\epsilon_B a_B\,. \tag{3.33}
$$

Since we assume that it vanishes for $A\neq B$, we obtain the crossing dimensions $\bar{q}_{AB}$ as

$$
\bar{q}_{AB}=\frac{(q_A+1)(q_B+1)}{D-2}-1-\frac{1}{2}\epsilon_A a_A\epsilon_B a_B\,. \tag{3.34}
$$

Eqs. (3.29) and (3.32) are then reduced to

$$
\sum_A\left[\frac{1}{2}M_{AA}H_A^2\tilde{E}_A^2-1\right]H_A^2(\partial\tilde{E}_A)^2=0\,, \tag{3.35}
$$

$$
\sum_A\left[\frac{1}{2}M_{AA}H_A^2\tilde{E}_A^2-1\right]H_A^2\left[\partial_i\tilde{E}_A\partial^j\tilde{E}_A-\frac{1}{d-1}(\partial\tilde{E}_A)^2\delta_i^j\right]=0\,. \tag{3.36}
$$

Hence, if

$$\frac{1}{2}M_{AA}H_A^2\tilde{E}_A^2 = 1, \quad \text{or} \quad \tilde{E}_A = \text{const}\,, \tag{3.37}$$

Eqs. (3.35) and (3.36) are satisfied.

Since

$$M_{AA} = \frac{(q_A+1)(D-q_A-3)}{D-2} + \frac{1}{2}a_A^2 \equiv \frac{\Delta_A}{D-2}\,, \tag{3.38}$$

from Eq. (3.37), we find $\tilde{E}_A$ as

$$\tilde{E}_A = \sqrt{\frac{2(D-2)}{\Delta_A}}\frac{1}{H_A}, \quad (\text{or} \quad \tilde{E}_A = \text{const})\,. \tag{3.39}$$

If we impose that a spacetime is asymptotically flat (i.e., $H_A \to 1$ as $r \to \infty$) and the potential E_A vanishes at infinity, we find that

$$E_A = -\sqrt{\frac{2(D-2)}{\Delta_A}}\left(1 - \frac{1}{H_A}\right), \quad (\text{or} \quad E_A = 0)\,. \tag{3.40}$$

Inserting this relation into Eq. (3.17), we obtain the equation for H_A as

$$\partial^2 H_A = 0\,, \tag{3.41}$$

which means that H_A is a harmonic function on $\{x^i\} \in \mathbb{E}^{d-1}$. From the relation (3.40) with Eqs. (3.12), (3.15) and (3.16), we then obtain the solutions for metric functions in terms of the harmonic functions H_A:

$$\begin{aligned} \xi &= -\sum_A \frac{D-q_A-3}{\Delta_A}\ln H_A, \quad \eta = \sum_A \frac{q_A+1}{\Delta_A}\ln H_A, \\ \zeta_\alpha &= -\sum_A \frac{\delta_{\alpha A}}{\Delta_A}\ln H_A, \quad \varphi = (D-2)\sum_A \frac{\epsilon_A a_A}{\Delta_A}\ln H_A\,. \end{aligned} \tag{3.42}$$

We have two remaining equations (3.13) and (3.18) for $\mathcal{A}_i$ ($\mathcal{F}_{ij}$) and one Poisson equation (3.11) for f . In order to solve the former two equations, we classify n_A-form field into the following three cases:

(1) Charged Branes

We expect that each brane A has a charge $\mathcal{Q}_H^{(A)}$ (either electric or magnetic type), and then E_A becomes non-trivial, i.e., $H_A \neq 1$. In this case, if we set

$$B_i^A = -\tilde{E}_A\mathcal{A}_i = -\sqrt{\frac{2(D-2)}{\Delta_A}}\frac{\mathcal{A}_i}{H_A}\,, \tag{3.43}$$

we have

$$\mathcal{F}_{ij}^{(A)} = \sqrt{\frac{2(D-2)}{\Delta_A}}\mathcal{F}_{ij} \,. \tag{3.44}$$

Inserting Eqs. (3.42), we can show that two equations (3.13) and (3.18) are reduced to the following one Laplace equation:

$$\partial^j \mathcal{F}_{ij} = 0 \,. \tag{3.45}$$

B_i^A describes a magnetic-type field produced by a current appearing through rotation of a charged brane.

It turns out that the condition (3.44) plays a key role for the system to keep Supersymmetry.

(2) Neutral Branes with Currents

If $H_A = 1$ (i.e., $E_A = 0$), which is a trivial solution of Eq. (3.41), we find that there is no electric type field, and then zero charge on the brane. This brane does not make any contribution to ξ, η, ζ_α and φ. Although the electric type field vanishes, B_i^A can still exist because a magnetic type field is produced by a curren, which is not charged. This current may appear in a system consisting of the same numbers of branes and anti-branes, which move with different velocities. This situation is similar to the conventional electric current in a metal. The negative charges of electrons balance with the positive ones of protons in the metal. The net charge vanishes, but a current is produced by the motion of electrons.

In this case ($H_A = 1$), the equations for $\mathcal{A}_i$ and B_i^A become independent as

$$\partial^j \mathcal{F}_{ij} = 0 \quad , \quad \partial^j \mathcal{F}_{ij}^{(A)} = 0 \,. \tag{3.46}$$

Since these two equations are exactly the same, we can adopt the same solution with different amplitudes, i.e.,

$$B_i^A = -\lambda_A \sqrt{\frac{2(D-2)}{\Delta_A}}\mathcal{A}_i \,, \;\; \mathcal{F}_{ij}^{(A)} = \lambda_A \sqrt{\frac{2(D-2)}{\Delta_A}}\mathcal{F}_{ij} \,, \tag{3.47}$$

where λ_A is an arbitrary constant, which corresponds to the strength of a current, i.e., numbers of branes and anti-branes and its relative velocity. This relation not only makes the equation for f simple (see below) but also keeps Supersymmetry of the system.

(3) Charged Branes with Currents

If numbers of branes and anti-branes are different, such a system has a net charge. Then the magnetic field may be divided into two parts:

$$B_i^A = -\sqrt{\frac{2(D-2)}{\Delta_A}}\frac{\mathcal{A}_i}{H_A} + B_i^{A(\mathcal{N})} \,, \tag{3.48}$$

where the first term is produced by a current appearing through the motion of a net charge, while $B_i^{A(\mathcal{N})}$ is produced by a current even in the case of zero net charge (just as in Case (2)). From Cases (1) and (2), we expect that $\mathcal{A}_i$ and $B_i^{A(\mathcal{N})}$ are arbitrary harmonic functions (just as Eqs. (3.46)). However, since we have two equations (3.13) and (3.18) for $\mathcal{A}_i$, it is not trivial whether our expectation is the case. In fact, we have to impose the following condition

$$\partial_{[i}B^A_{j]} \cdot \partial^j \tilde{E}_A = 0 \ , \tag{3.49}$$

in order for $\mathcal{A}_i$ and B_i^A to be a solution.

If we can impose the relation of $B_i^{A(\mathcal{N})} \propto \mathcal{A}_i$ just as that for B_i^A in Case (2), the equation for f becomes simple, but this relation is not consistent with the condition (3.79). As a result, the equation for f becomes very complicated, although we can solve it in principle because it is the Poisson equation in a flat space. We also find that this condition breaks the Supersymmetry of the system. Therefore, in what follows, we will discuss only Cases of (1) and (2).

Finally we discuss the last equation (3.11) for f. Here we assume we have $N_{A'}$ charged branes (Case (1)) and $N_{A''}$ neutral branes with currents (Case (2)). $N_A = N_{A'} + N_{A''}$ is the total number of branes. Then, as for the metric f, we find

$$\partial^2 f = \frac{\beta}{2}\prod_A H_A^{-\frac{2(D-2)}{\Delta_A}} \left(\partial_{[j}\mathcal{A}_{i]}\right)^2 \ , \tag{3.50}$$

where

$$\beta = \left[1-(D-2)\left(\sum_{A'}\frac{1}{\Delta_{A'}} + \sum_{A''}\frac{\lambda^2_{A''}}{\Delta_{A''}}\right)\right] \ , \tag{3.51}$$

is just a constant. A' describes charged branes which provide non-trivial potentials E_A (3.39), while A'' gives a contribution from the neutral branes with currents.

If the following condition is satisfied:

$$(D-2)\left(\sum_{A'}\frac{1}{\Delta_{A'}} + \sum_{A''}\frac{\lambda^2_{A''}}{\Delta_{A''}}\right) = 1 \,, \tag{3.52}$$

β vanishes, and then f is given by an arbitrary harmonic function on $\mathbb{E}^{d-1}$. In general, however, since $\lambda_{A''}$ is free, β can have any sign (either positive, zero or negative). Thus, we have to solve the Poisson equation (3.50).

The solution obtained in this section is summarized as follows:

$$\begin{aligned} ds^2 &= \prod_A H_A^{2\frac{q_A+1}{\Delta_A}}\left[2\prod_B H_B^{-2\frac{D-2}{\Delta_B}} du\left(dv + f du + \frac{\mathcal{A}}{\sqrt{2}}\right) + \sum_{\alpha=2}^{p}\prod_B H_B^{-2\frac{\gamma_{\alpha B}}{\Delta_B}} dy_\alpha^2 + \sum_{i=1}^{d-1} dx_i^2\right] , \\ \varphi &= (D-2)\sum_A \frac{\epsilon_A a_A}{\Delta_A}\ln H_A \,, \end{aligned} \tag{3.53}$$

where

$$\begin{aligned}
\Delta_A &= (q_A+1)(D-q_A-3)+\frac{D-2}{2}a_A^2, \\
\gamma_{\alpha A} &= \delta_{\alpha A}+q_A+1 = \begin{cases} D-2 & \alpha=\alpha_2,\cdots,\alpha_{q_A} \\ 0 & \text{otherwise} \end{cases} .
\end{aligned} \tag{3.54}$$

H_A for each q_A-brane and $\mathcal{A} = \mathcal{A}_i dx^i$ are arbitrary harmonic functions, while the vector potential B_i^A can be chosen either as $B_i^A \propto \mathcal{A}_i/H_A$ (when $H_A \neq 1$), or an arbitrary harmonic function (when $H_A = 1$). The wave metric f usually satisfies the Poisson equation (3.50) with some source term originated by the rotation-induced metric $\mathcal{A}_i$, although it can be also an arbitrary harmonic function for some specific configuration of branes ($\beta = 0$)[52].

It is worth noticing that we have independent Laplace equations for H_A and $\mathcal{A}_i$ (and f when $\beta = 0$). This makes the construction of solutions very easy. The superposition of any solutions also provides us an exact solution. Hence we can construct an infinite number of solutions. We can also show that a part of Supersymmetry is preserved in Cases (1) and (2) if $\mathcal{F}_{ij}$ is self-dual.

3.1.2 M2 and M5-brane Solutions in M-theory

In 11-dimensional Supergravity, we have a 4-form field ($n_A = 4$) and no dilaton φ ($a_A = 0$). Setting $D = 11$ and $a_A = 0$, we have

$$\Delta_A = (q_A+1)(8-q_A)\,. \tag{3.55}$$

The form field produced by an electric charge is related to the M2-brane, i.e., $q_A = n_A - 2 = 2$. This gives $\Delta_A = 18$. The black brane solution in this case is written as

$$\begin{aligned}
ds_{11}^2 &= H_2^{1/3}\left[2H_2^{-1}du\left(dv+fdu+\frac{\mathcal{A}}{\sqrt{2}}\right)+H_2^{-1}dy_6^2+\sum_{i=1}^{8}dx_i^2\right], \\
F_4 &= d(1/H_2)\wedge du\wedge dv\wedge dy_6+\frac{1}{\sqrt{2}}dB_2\wedge du\wedge dy_6\,,
\end{aligned} \tag{3.56}$$

where H_2 is a harmonic function on $\mathbb{E}^8$.

Similarly, the field with a magnetic charge is related to the M5-brane because $*q_A = *n_A - 2 = D - n_A - 2 = 5$. This also gives $\Delta_A = 18$. The solution is described by

$$\begin{aligned}
ds_{11}^2 &= H_5^{2/3}\left[2H_5^{-1}du\left(dv+fdu+\frac{\mathcal{A}}{\sqrt{2}}\right)+H_5^{-1}\sum_{\alpha=2}^{5}dy_\alpha^2+\sum_{i=1}^{5}dx_i^2\right], \\
*F_4 &= d(1/H_5)\wedge du\wedge dv\wedge dy_2\wedge dy_3\wedge dy_4\wedge dy_5 \\
&\quad +\frac{1}{\sqrt{2}}dB_5\wedge du\wedge dy_2\wedge dy_3\wedge dy_4\wedge dy_5\,,
\end{aligned} \tag{3.57}$$

where H_5 is a harmonic function on $\mathbb{E}^5$. In both cases, $\mathcal{A}_i$ is also a vector harmonic function, while f is given by the Poisson equation (3.50) with $\beta = 1/2$ because of Eq. (3.51).

These two branes (M2 and M5) can intersect if and only if

$$M2 \cap M2 \to \bar{q}_{22} = 0, \quad M2 \cap M5 \to \bar{q}_{25} = 1, \quad M5 \cap M5 \to \bar{q}_{55} = 3\,. \tag{3.58}$$

The crossing rule leads that there exists a four-dimensional (4D) black object with four independent branes (or three M5 branes and one wave), or a five-dimensional (5D) black object with three independent M2 branes (or two branes and one wave) (see Table 3.1). The 4 D black object with M2⊥M2⊥M5⊥M5 branes and the 5D object with M2⊥M2⊥M2 branes have no traveling wave. While, the 4D black object with M5⊥M5⊥M5⊥W branes and the 5D object with M2⊥M5⊥W branes describe stationary spacetimes with a traveling wave. We shall discuss the details of the 5D black object with M2⊥M5⊥W branes in the next section.

$d=4$

y_1	y_2	y_3	y_4	y_5	y_6	y_7
M2	M2					
		M2	M2			
M5		M5		M5	M5	M5
	M5		M5	M5	M5	M5

$d=4$

y_1	y_2	y_3	y_4	y_5	y_6	y_7
M5			M5	M5	M5	M5
M5	M5	M5			M5	M5
M5	M5	M5	M5	M5		
W						

$d=5$

y_1	y_2	y_3	y_4	y_5	y_6
M2	M2				
		M2	M2		
				M2	M2

$d=5$

y_1	y_2	y_3	y_4	y_5	y_6
M2					M2
M5	M5	M5	M5	M5	
W					

Table 3.1: Some examples of intersecting branes for $d = 4$ and 5. M2, M5 and W denote the location where the M2 brane, the M5 brane and a wave exist, respectively.

We can also consider the case of $\mathcal{N} = 2, D = 10$ type IIB Supergravity theory, and the action in the Einstein frame is given by

$$S = \frac{1}{2\kappa^2}\int d^{10}X\sqrt{-g}\left[\mathcal{R} - \frac{1}{2}(\nabla\varphi)^2 - \frac{1}{2\cdot n_A!}e^{\frac{5-n_A}{2}\varphi}F^2_{\mathbf{n_A}}\right]. \tag{3.59}$$

The coupling constant a_A in the previous action (3.1) is given by $a_A = (5 - n_A)/2$. The three-form field with an electric charge is related to the D1-brane, i.e., $q_A = n_A - 2 = 1$. Then we find $\Delta_A = (q_A + 1)(7 - q_A) + 4a_A^2 = 16$, which does not depend on the type of branes (A or n_A). This gives the same value of $(D - 2)/\Delta_A = 1/2$ as that in the case of eleven-dimensional Supergravity. Then we find solutions in type IIB Supergravity similar to those in eleven-dimensional Supergravity. In fact a black brane solution in this case is written as

$$ds_{10}^2 = H_1^{1/4}\left[2H_1^{-1}du\left(dv + fdu + \frac{\mathcal{A}}{\sqrt{2}}\right) + \sum_{i=1}^{8} dx_i^2\right] \quad , \tag{3.60}$$

where H_1 is a harmonic function on $\mathbb{E}^8$. The form field with a magnetic charge is related to the D5-brane because $*q_A = *n_A - 2 = D - n_A - 2 = 5$. This also gives $\Delta_A = 16$. The solutions is described by

$$ds_{10}^2 = H_5^{-1/4}\left[2du\left(dv + fdu + \frac{\mathcal{A}}{\sqrt{2}}\right) + H_5\sum_{i=1}^{4} dx_i^2 + \sum_{\alpha=2}^{5} dy_\alpha^2\right] , \tag{3.61}$$

where H_5 is a harmonic function on $\mathbb{E}^4$. The solutions with two intersecting branes (D1 and D5) are also given just as the previous subsection.

3.1.3 Supersymmetry in M2⊥M5 Black Branes

Here we discuss Supersymmetry in the solution obtained in this paper. The invariance for Supersymmetry transformation of a gravitino gives a criterion for existence of unbroken Supersymmetry. This condition is given by the Killing equation for the Killing spinor ϵ [53], i.e.,

$$\delta\psi_{\hat{a}} = \left[e_{\hat{a}}^{\ \mu}\partial_\mu + \frac{1}{4}w^{\hat{b}\hat{c}}{}_{\hat{a}}\gamma_{\hat{b}\hat{c}} + \frac{1}{288}(\gamma_{\hat{a}}{}^{\hat{b}\hat{c}\hat{d}\hat{e}} - 8\delta_{\hat{a}}^{\ \hat{b}}\gamma^{\hat{c}\hat{d}\hat{e}})F_{\hat{b}\hat{c}\hat{d}\hat{e}}\right]\epsilon = 0\,, \tag{3.62}$$

where $\gamma_{\hat{a}\hat{b}}$'s are the antisymmetrized products of eleven-dimensional gamma matrices with unit strength on vielbein $e^{\hat{a}}{}_{\mu}$, and a spin connection is given by

$$w^{\hat{b}\hat{c}}{}_{\hat{a}} = e_{\hat{a}}^{\ \mu}\left(e^{\hat{b}\nu}\partial_{[\mu}e^{\hat{c}}{}_{\nu]} - e^{\hat{c}\nu}\partial_{[\mu}e^{\hat{b}}{}_{\nu]} - e^{\hat{b}\rho}e^{\hat{c}\sigma}e_{\hat{d}\mu}\partial_{[\rho}e^{\hat{d}}{}_{\sigma]}\right) . \tag{3.63}$$

Now we consider the M2⊥M5 black brane solution related to the five-dimensional black hole, and the metric for the space with M2⊥M5 intersecting branes are given by

$$ds^2 = H_2^{1/3}H_5^{2/3}\left[2(H_2H_5)^{-1}du\left(dv + fdu + \frac{\mathcal{A}}{\sqrt{2}}\right) + \sum_{\alpha=2}^{5} H_5^{-1}dy_\alpha^2 + H_2^{-1}dy_6^2 + \sum_{i=1}^{4} dx_i^2\right] \tag{3.64}$$

The non-trivial components of field strength $F_{\hat{a}\hat{b}\hat{c}\hat{d}}$ are given by

$$F_{\hat{j}\hat{u}\hat{v}\hat{y}_6} = -H_2^{-1/6}H_5^{-1/3}\frac{\partial_j H_2}{H_2} \tag{3.65}$$

$$F_{\hat{i}\hat{j}\hat{u}\hat{y}_6} = -\frac{1}{\sqrt{2}}H_2^{-2/3}H_5^{-5/6}\left(\mathcal{F}^{(2)}_{\hat{i}\hat{j}} + *\mathcal{F}^{(5)}_{\hat{i}\hat{j}}\right) \tag{3.66}$$

$$F_{\hat{k}\hat{l}\hat{m}\hat{y}_6} = \epsilon^{jklm}H_2^{-1/6}H_5^{-1/3}\frac{\partial_j H_5}{H_5}\,, \tag{3.67}$$

where $\mathcal{F}^{(A)}_{\hat{i}\hat{j}}$ and those duals $*\mathcal{F}^{\hat{i}\hat{j}}_{(A)}$ are given by

$$\begin{aligned}\mathcal{F}^{(A)}_{\hat{i}\hat{j}} &\equiv -2H_A\left(\partial_{[i}B^{(A)}_{j]} - \mathcal{A}_{[i}\partial_{j]}E_A\right)\\ &= -2H_A\left(\partial_{[i}B^{(A)}_{j]} + \frac{1}{H_A^2}\mathcal{A}_{[i}\partial_{j]}H_A\right) \qquad (3.68)\\ *\mathcal{F}^{\hat{i}\hat{j}}_{(A)} &\equiv \frac{1}{2}\epsilon^{ijkl}\mathcal{F}^{(A)}_{kl}\,. \qquad (3.69)\end{aligned}$$

If we have two charged branes (Case (1): $B^{(A)}_i = -\mathcal{A}_i/H_A$), each $\mathcal{F}^{(A)}_{\hat{i}\hat{j}}$ coincides with $\mathcal{F}_{\hat{i}\hat{j}}$. For neutral branes with currents (Case (2)), we find $\mathcal{F}^{(A)}_{\hat{i}\hat{j}} = \lambda_A\mathcal{F}_{\hat{i}\hat{j}}$.

Using Eq. (3.40) and the above explicit expression for $F_{\hat{a}\hat{b}\hat{c}\hat{d}}$, we obtain the Killing equations (3.62) as

$$\begin{aligned}\delta\psi_{\hat{u}} &= \frac{1}{6}H_2^{-1/6}H_5^{-1/3}\left[\frac{\partial_j H_2}{H_2}\gamma^{\hat{j}\hat{v}}(1-\gamma^{\hat{u}\hat{v}\hat{y}_6}) + \frac{1}{2}\frac{\partial_j H_5}{H_5}\gamma^{\hat{j}\hat{v}}(1-\gamma^{\hat{u}\hat{v}\hat{y}_2\cdots\hat{y}_5})\right]\epsilon\\ &\quad -\frac{1}{4\sqrt{2}}H_2^{-2/3}H_5^{-5/6}\left[\left\{\frac{1}{2}\mathcal{F}_{\hat{i}\hat{j}} + \frac{1}{3}\left(*\mathcal{F}^{(2)}_{\hat{i}\hat{j}} + \mathcal{F}^{(5)}_{\hat{i}\hat{j}}\right) - \frac{1}{6}\left(\mathcal{F}^{(2)}_{\hat{i}\hat{j}} + *\mathcal{F}^{(5)}_{\hat{i}\hat{j}}\right)\right\}\gamma^{\hat{i}\hat{j}}\right.\\ &\quad \left. -\frac{1}{3}\left(*\mathcal{F}^{(2)}_{\hat{i}\hat{j}} + \mathcal{F}^{(5)}_{\hat{i}\hat{j}}\right)\gamma^{\hat{i}\hat{j}}(1-\gamma^{\hat{u}\hat{v}\hat{y}_2\cdots\hat{y}_5}) + \frac{1}{6}\left(\mathcal{F}^{(2)}_{\hat{i}\hat{j}} + *\mathcal{F}^{(5)}_{\hat{i}\hat{j}}\right)\gamma^{\hat{i}\hat{j}}(1-\gamma^{\hat{u}\hat{v}\hat{y}_6})\right]\epsilon\\ &\quad -\frac{1}{2}H_2^{-1/6}H_5^{-1/3}\partial^j f\gamma^{\hat{j}\hat{u}}\epsilon\,, \qquad (3.70)\end{aligned}$$

$$\delta\psi_{\hat{v}} = \frac{1}{6}H_2^{-1/6}H_5^{-1/3}\left[\frac{\partial_j H_2}{H_2}\gamma^{\hat{j}\hat{u}}(1-\gamma^{\hat{u}\hat{v}\hat{y}_6}) + \frac{\partial_j H_5}{2H_5}\gamma^{\hat{j}\hat{u}}(1-\gamma^{\hat{u}\hat{v}\hat{y}_2\cdots\hat{y}_5})\right]\epsilon\,, \qquad (3.71)$$

$$\begin{aligned}\delta\psi_{\hat{i}} &= H_2^{-1/6}H_5^{-1/3}\left(\partial_i + \frac{\partial_i H_2}{6H_2} + \frac{\partial_i H_5}{12H_5}\right)\epsilon\\ &\quad +\frac{1}{\sqrt{2}}H_2^{-2/3}H_5^{-5/6}\left[\left\{\frac{1}{4}\mathcal{F}_{\hat{i}\hat{j}} - \frac{1}{6}\left(\mathcal{F}^{(2)}_{\hat{i}\hat{j}} + *\mathcal{F}^{(5)}_{\hat{i}\hat{j}}\right) - \frac{1}{12}\left(*\mathcal{F}^{(2)}_{\hat{i}\hat{j}} + \mathcal{F}^{(5)}_{\hat{i}\hat{j}}\right)\right\}\gamma^{\hat{j}\hat{u}}\right.\\ &\quad \left. +\frac{1}{6}\left(\mathcal{F}^{(2)}_{\hat{i}\hat{j}} + *\mathcal{F}^{(5)}_{\hat{i}\hat{j}}\right)\gamma^{\hat{j}\hat{u}}(1-\gamma^{\hat{u}\hat{v}\hat{y}_6}) + \frac{1}{12}\left(*\mathcal{F}^{(2)}_{\hat{i}\hat{j}} + \mathcal{F}^{(5)}_{\hat{i}\hat{j}}\right)\gamma^{\hat{j}\hat{u}}(1-\gamma^{\hat{u}\hat{v}\hat{y}_2\cdots\hat{y}_5})\right]\epsilon\\ &\quad +\frac{1}{6}H_2^{-1/6}H_5^{-1/3}\left[\frac{\partial_j H_2}{2H_2}\gamma^{\hat{i}\hat{j}}(1-\gamma^{\hat{u}\hat{v}\hat{y}_6}) + \frac{\partial^j H_5}{H_5}\gamma^{\hat{i}\hat{j}}(1-\gamma^{\hat{u}\hat{v}\hat{y}_2\cdots\hat{y}_5})\right]\epsilon\\ &\quad -\frac{1}{6}H_2^{-1/6}H_5^{-1/3}\left[\frac{\partial_i H_2}{H_2}(1-\gamma^{\hat{u}\hat{v}\hat{y}_6}) + \frac{\partial_i H_5}{2H_5}(1-\gamma^{\hat{u}\hat{v}\hat{y}_2\cdots\hat{y}_5})\right]\epsilon\,, \qquad (3.72)\end{aligned}$$

$$\begin{aligned}\delta\psi_{\hat{y}_{2(\cdots5)}} &= -\frac{1}{12}H_2^{-1/6}H_5^{-1/3}\left[\frac{\partial_j H_2}{H_2}\gamma^{\hat{j}\hat{y}_{2(\cdots5)}}(1-\gamma^{\hat{u}\hat{v}\hat{y}_6}) - \frac{\partial_j H_5}{H_5}\gamma^{\hat{j}\hat{y}_{2(\cdots5)}}(1-\gamma^{\hat{u}\hat{v}\hat{y}_2\cdots\hat{y}_5})\right]\epsilon\\ &\quad +\frac{1}{24\sqrt{2}}H_2^{-2/3}H_5^{-5/6}\left(\mathcal{F}^{(2)}_{\hat{i}\hat{j}} + *\mathcal{F}^{(5)}_{\hat{i}\hat{j}}\right)\gamma^{\hat{i}\hat{j}\hat{u}\hat{y}_{2(\cdots5)}\hat{y}_6}\epsilon\,, \qquad (3.73)\end{aligned}$$

$$\begin{aligned}\delta\psi_{\hat{y}_6} &= \frac{1}{6}H_2^{-1/6}H_5^{-1/3}\left[\frac{\partial_j H_2}{H_2}\gamma^{\hat{j}\hat{y}_6}(1-\gamma^{\hat{u}\hat{v}\hat{y}_6}) - \frac{\partial_j H_5}{H_5}\gamma^{\hat{j}\hat{y}_6}(1-\gamma^{\hat{u}\hat{v}\hat{y}_2\cdots\hat{y}_5})\right]\epsilon \\ &\quad -\frac{1}{12\sqrt{2}}H_2^{-2/3}H_5^{-5/6}\left(\mathcal{F}^{(2)}_{\hat{i}\hat{j}} + *\mathcal{F}^{(5)}_{\hat{i}\hat{j}}\right)\gamma^{\hat{i}\hat{j}\hat{u}}\epsilon\,. \end{aligned} \tag{3.74}$$

Most parts of the above equations vanish if we impose the following condition for the Killing spinor ϵ:

$$(1-\gamma^{\hat{u}\hat{v}\hat{y}_6})\epsilon = 0,\ (1-\gamma^{\hat{u}\hat{v}\hat{y}_2\cdots\hat{y}_5})\epsilon = 0,\ \gamma^{\hat{u}}\epsilon = 0\,. \tag{3.75}$$

These conditions can be rewritten as

$$(1+\gamma^{\hat{0}\hat{y}_1\hat{y}_6})\epsilon = 0, \quad (1+\gamma^{\hat{0}\hat{y}_1\cdots\hat{y}_5})\epsilon = 0, \quad (1+\gamma^{\hat{0}\hat{y}_1})\epsilon = 0\,. \tag{3.76}$$

However, in order to satisfy the Supersymmetric condition $\delta\psi = 0$, there are two terms remain;

$$\left(\partial_i + \frac{\partial_i H_2}{6H_2} + \frac{\partial_i H_5}{12H_5}\right)\epsilon\,, \tag{3.77}$$

$$\left\{\frac{1}{2}\mathcal{F}_{\hat{i}\hat{j}} + \frac{1}{3}\left(*\mathcal{F}^{(2)}_{\hat{i}\hat{j}} + \mathcal{F}^{(5)}_{\hat{i}\hat{j}}\right) - \frac{1}{6}\left(\mathcal{F}^{(2)}_{\hat{i}\hat{j}} + *\mathcal{F}^{(5)}_{\hat{i}\hat{j}}\right)\right\}\gamma^{\hat{i}\hat{j}}\epsilon\,. \tag{3.78}$$

The former term (3.77) vanishes if ϵ is described as

$$\epsilon = H_2^{-1/6}H_5^{-1/12}\epsilon_0\,, \tag{3.79}$$

where ϵ_0 is a constant spinor. The latter term also vanishes if $\mathcal{F}^{(A)}_{ij} \propto \mathcal{F}_{ij}$ $(A = 2, 5)$ and $\mathcal{F}_{ij}$ is self-dual ($*\mathcal{F}_{ij} = \mathcal{F}_{ij}$). In fact, this term is proportional to $\mathcal{F}_{\hat{i}\hat{j}}\gamma^{\hat{i}\hat{j}}\epsilon$ which vanishes for the self-dual field $\mathcal{F}_{ij}$ as shown below. From Eqs. (3.76), we have

$$\gamma^{\hat{0}\hat{y}_1}\epsilon = -\epsilon\,, \quad \gamma^{\hat{y}_6}\epsilon = \epsilon\,, \quad \gamma^{\hat{y}_2\cdots\hat{y}_5}\epsilon = \epsilon\,. \tag{3.80}$$

We also assume that $\gamma^{\hat{0}y_1\cdots\hat{y}_6\hat{x}_1\hat{x}_2\hat{x}_3\hat{x}_4}\epsilon = -\epsilon$ which corresponds to the chiral state. Then, we find $\gamma^{\hat{x}_1\hat{x}_2\hat{x}_3\hat{x}_4}\epsilon = \epsilon$, and it can be rewritten as $\gamma^{\hat{i}\hat{j}}\epsilon = -\epsilon_{\hat{i}\hat{j}\hat{k}\hat{l}}/2\gamma^{\hat{k}\hat{l}}\epsilon$. We then obtain

$$\mathcal{F}_{ij}\gamma^{\hat{i}\hat{j}}\epsilon = -\frac{1}{2}\mathcal{F}_{ij}\epsilon_{\hat{i}\hat{j}\hat{k}\hat{l}}\gamma^{\hat{k}\hat{l}}\epsilon = -*\mathcal{F}_{kl}\gamma^{\hat{k}\hat{l}}\epsilon\,, = -\mathcal{F}_{kl}\gamma^{\hat{k}\hat{l}}\epsilon\,. \tag{3.81}$$

The last equality is found by the self-duality of $\mathcal{F}_{ij}$. This equation yields

$$\mathcal{F}_{ij}\gamma^{\hat{i}\hat{j}}\epsilon = 0\,. \tag{3.82}$$

In the case of the BMPV type solution discussed in §IV B, this self-dual condition gives the relation between J_ϕ and J_ψ, that is, $J_\phi = -J_\psi = J$.

The condition of $\mathcal{F}^{(A)}_{ij} \propto \mathcal{F}_{ij}$ leads either to Case (1):two charged branes, or Case (2): neutral branes with currents discussed in §II. We conclude that the solutions discussed in this paper preserve $1/8$ Supersymmetry if $\mathcal{F}_{ij}$ is self-dual [54].

We also expect that the Kalza-Klein compactification into five-dimensional spacetime does not break any Supersymmetry, because all coordinates to be compactified are cyclic. Thus the five dimensional solution obtained here is a BPS state if $\mathcal{F}_{ij}$ is self-dual.

3.2 Null Dependent Brane Solutions

In String Theory, D-brane which is the end of the open strings, is derived from the Dirichlet boundary condition for time-like direction. However it is possible for considering the Dirichlet boundary condition for space-like direction or null direction, which are called S-brane given by [55] or N-brane given by [56]. Until now, even though time-dependent brane solutions can be considered in both cases, relatively, S-brane solutions acquire much more attention because of their possible connection with rolling tachyon and dS/CFT correspondence [57, 55, 58, 59, 60, 61, 62, 63, 64] as well as inflationary solutions [65, 66, 67, 68, 69] (see also Refs. [70, 71, 72, 73, 74] for related solutions). As in D-brane cases, intersecting S-brane solutions can be also obtained [75, 76]. On the other hand, N-brane solutions are also interesting from the viewpoint of closed/open string correspondence and stringy explanation of the black holes as discussed in Ref. [56], where such solutions were discussed in the string worldsheet picture. Recently, some class of explicit intersecting N-brane solutions in supergravity have been obtained [76] in which the intersection rules for the way the solutions can intersect with each other is given based on the method of [77, 78]. The main purpose of this paper is to construct on other class of N-brane cosmological solutions that are reminiscent of a stringy set-up.

3.2.1 Construction to Null Brane Solutions

We take the metric ansatz for N-branes as

$$ds^2 = 2\theta^{\hat{u}}\theta^{\hat{v}} + \sum_{i=1}^{d-1}\left(\theta^{\hat{i}}\right)^2 + \sum_{\alpha=2}^{p}\left(\theta^{\hat{\alpha}}\right)^2 , \tag{3.83}$$

where the dual basis $\theta^{\hat{a}}$ are given by

$$\theta^{\hat{u}} = e^{\xi}du\,, \quad \theta^{\hat{v}} = e^{\xi}(dv + f du)\,, \quad \theta^{\hat{i}} = e^{\eta}dx^i\,, \quad \theta^{\hat{\alpha}} = e^{\alpha dy^{\alpha}}\,. \tag{3.84}$$

Here D-dimensional manifold can be represented by the light-cone coordinate $u = -(t - y_1)/\sqrt{2}$ and $v = (t + y_1)/\sqrt{2}$. Furthermore, as for the number of the spacetime dimensions, we separate as a compact $p-1$-dimensional Euclidian space with coordinate y_α $(\alpha = 2, ..., p)$ and non-compact $d-1$-dimensional Euclidian space with coordinate x^i $(i = 1, \ldots d)$. In order to find null-dependent solutions, all the metric functions and the dilatonφ are depend only on the light-cone coordinate u and v. These solutions about the light cone coordinate are named null-brane (N-branes), and N$_{q_A}$-brane whose world-volume is (q_A+1)- dimensional, tangential to u, and q_A-spacelike directions.

As for the n_A-form fields with a q_A-brane, we assume that the souse brane exists in the coordinate $\{y_1, \ldots, y_{q_A+1}\}$. The form field generated by an electric charge is given by

$$F_{\mathbf{n_A}} = (\partial_u E_A - f\partial_v E_A)\, du \wedge dv \wedge dy_2 \wedge \cdots \wedge dy_{q_A+1}\,, \tag{3.85}$$

where $n_A = q_A + 2$ and E_A is a scalar potential which depends on u and v. This setting automatically guarantees the Bianchi identity (3.4).

We can also discuss the form field generated by a magnetic charge by use of a dual $*n_A$-form field with $*q_A$-brane, which is obtained by a dual transformation of the n_A-field with a q_A-brane ($*n_A \equiv D - n_A$, $*q_A \equiv *n_A - 2$). In other words, the field components of $F_{\mathbf{n_A}}$ generated by a magnetic charge are described by the same form of (3.85) of the dual field $*F_{\mathbf{n_A}} = F_{*\mathbf{n_A}}$. We then treat $F_{*\mathbf{n_A}}$, which is generated by a magnetic charge, as another independent form field with a different brane from $F_{\mathbf{n_A}}$, which is generated by an electric charge, when we sum up by the types of branes A.

Now we setting

$$H_A = \exp\left[-\left(\xi + \sum_{\alpha=2}^{q_A+1}\zeta_\alpha - \frac{1}{2}\epsilon_A a_A \varphi\right)\right] \tag{3.86}$$

$$V = \exp\left[(d-1)\eta + \sum_{\alpha=2}^{p}\zeta_\alpha\right], \tag{3.87}$$

where ϵ_A is related to the electric and magnetic sign as

$$\epsilon_A = \begin{cases} +1 & n_A\text{-form field } (F_{\mathbf{n_A}}) \\ -1 & \text{the dual field } (*F_{\mathbf{n_A}}) \end{cases}, \tag{3.88}$$

To simplify we choose the gauge condition as $V = 1$ and then the basic equations can be expressed as follows;

$$(d-1)(\partial_u\eta - f\partial_v\eta)^2 + \sum_\alpha(\partial_u\zeta_\alpha - f\partial_v\zeta_\alpha)^2 = -\frac{1}{2}(\partial_u\varphi - f\partial_v\varphi)^2 \tag{3.89}$$

$$\begin{aligned} &2\partial_v(\partial_u\xi - f\partial_v\xi) - \partial_v^2 f + (d-1)(\partial_u\eta - f\partial_v\eta)\partial_v\eta + \sum_\alpha(\partial_u\zeta_\alpha - f\partial_v\zeta_\alpha)\partial_v\zeta_\alpha \\ &\quad = -\frac{1}{2}(\partial_u\varphi - f\partial_v\varphi)\partial_v\varphi + \sum_A\frac{D-q_A-3}{2(D-2)}2H_A^2(\partial_u E_A - f\partial_v E_A)^2 \end{aligned} \tag{3.90}$$

$$(d-1)(\partial_v\eta)^2 + \sum_\alpha(\partial_v\zeta_\alpha)^2 = -\frac{1}{2}(\partial_v\varphi)^2 \tag{3.91}$$

$$\partial_v(\partial_u\eta - f\partial_v\eta) = -\sum_A\frac{q_A+1}{2(D-2)}H_A^2(\partial_u E_A - f\partial_v E_A)^2 \tag{3.92}$$

$$\partial_v(\partial_u\zeta_\alpha - f\partial_v\zeta_\alpha) = \sum_A\frac{\delta_{\alpha A}}{2(D-2)}H_A^2(\partial_u E_A - f\partial_v E_A)^2 \tag{3.93}$$

$$\partial_v(\partial_u\varphi - f\partial_v\varphi) = -\sum_A\frac{1}{2}\epsilon_A a_A H_A^2(\partial_u E_A - f\partial_v E_A)^2 \tag{3.94}$$

$$\partial_u\left[H_A^2(\partial_u E_A - f\partial_v E_A)\right] = \partial_v\left[H_A^2(\partial_u E_A - f\partial_v E_A)\right] = 0\,, \tag{3.95}$$

where $\delta_{\alpha A}$ for each coordinate α are defined by

$$\delta_{\alpha A} = \begin{cases} D - q_A - 3 & y_\alpha \text{ belonging to } q_A\text{-brane} \\ -(q_A + 1) & \text{otherwise} \end{cases} . \tag{3.96}$$

In this equations we much add the integrable condition as $\partial_v f = 0$ and $\partial_u E_A = 0$. Using these condition and Eqs.(3.95), we can integrate as

$$\partial_v \left[(\partial_u - f\partial_v)\xi + \sum_A \frac{D - q_A - 3}{2(D-2)} f(\partial_u E_A - f\partial_v E_A) H_A^2 E_A \right] = 0 \tag{3.97}$$

$$\partial_v \left[(\partial_u - f\partial_v)\eta - \sum_A \frac{q_A + 1}{2(D-2)} f(\partial_u E_A - f\partial_v E_A) H_A^2 E_A \right] = 0 \tag{3.98}$$

$$\partial_v \left[(\partial_u - f\partial_v)\zeta_\alpha + \sum_A \frac{\delta_{\alpha A}}{2(D-2)} f(\partial_u E_A - f\partial_v E_A) H_A^2 E_A \right] = 0 \tag{3.99}$$

$$\partial_v \left[(\partial_u - f\partial_v)\varphi - \sum_A \frac{1}{2} \epsilon_A a_A f(\partial_u E_A - f\partial_v E_A) H_A^2 E_A \right] = 0\,. \tag{3.100}$$

In this paper, we concentrate on the solutions with the following conditions, which satisfy the above equations automatically;

$$(\partial_u - f\partial_v)\xi = -\sum_A \frac{D - q_A - 3}{2(D-2)} f(\partial_u E_A - f\partial_v E_A) H_A^2 E_A \tag{3.101}$$

$$(\partial_u - f\partial_v)\eta = \sum_A \frac{q_A + 1}{2(D-2)} f(\partial_u E_A - f\partial_v E_A) H_A^2 E_A \tag{3.102}$$

$$(\partial_u - f\partial_v)\zeta_\alpha = -\sum_A \frac{\delta_{\alpha A}}{2(D-2)} f(\partial_u E_A - f\partial_v E_A) H_A^2 E_A \tag{3.103}$$

$$(\partial_u - f\partial_v)\varphi = \sum_A \frac{1}{2} \epsilon_A a_A f(\partial_u E_A - f\partial_v E_A) H_A^2 E_A\,. \tag{3.104}$$

We show these conditions are consistent with the BPS condition, of an extremal solution in Supergravity. Eqs.(3.89) and (3.91) can be rewritten as

$$M_{AB} = (d-1)\frac{(q_A + 1)(q_B + 1)}{(D-2)^2} + \frac{1}{2}\epsilon_A a_A \epsilon_B a_B \sum_{\alpha=2}^{p} \frac{\delta_{\alpha A}\delta_{\alpha B}}{(D-2)^2} = 0\,. \tag{3.105}$$

Suppose that Nq_A-brane and Nq_B-brane intersect over $\bar{q}_{AB} + 1$ dimensions $(\bar{q}_{AB} < q_A, q_B)$ a rule for the crossing dimensions, which is called the crossing rule of the branes is obtained as

$$\bar{q}_{AB} = \frac{(q_A + 1)(q_B + 1)}{D - 2} - 1 - \frac{1}{2}\epsilon_A a_A \epsilon_B a_B\,. \tag{3.106}$$

This crossing rule is the same as that for the S-brane cases given by [75][76].

From the consistency condition, the components of metric ξ, η, and ζ_α is only depend on the v direction, which is also consistent to the Supersymmety condition we consider in below. Setting

$$f(u)E_A(v) = \frac{1}{H(u,v)}\,, \tag{3.107}$$

the Eqs (3.18) can be changed by

$$\partial_v^2 H_A = \partial_v\partial_u H_A = 0\,. \tag{3.108}$$

Therefore H_A can be solved as

$$H_A = \alpha v + g(u)\,, \tag{3.109}$$

where α is a constant, and $g(u)$ is an arbitrary function of u.

The basic equations finally becomes

$$\partial_v\xi = \sum_A \frac{D-q_A-3}{2(D-2)}\partial_v \ln H_A \tag{3.110}$$

$$\partial_v\eta = -\sum_A \frac{q_A+1}{2(D-2)}\partial_v \ln H_A \tag{3.111}$$

$$\partial_v\zeta_\alpha = \sum_A \frac{\delta_{\alpha A}}{2(D-2)}\partial_v \ln H_A \tag{3.112}$$

$$\partial_v\varphi = -\sum_A \frac{1}{2}\epsilon_A a_A \partial_v \ln H_A\,. \tag{3.113}$$

Therefore we can also integrate these equations, and we find the solutions as

$$ds^2 = \prod_A H_A^{\frac{q_A+1}{D-2}}\left[\prod_A H_A^{-1} 2du(dv+fdu) + \sum_{\alpha=2}^{p}\prod_A H_A^{\frac{-\gamma_A}{D-2}} dy_\alpha^2 + \sum_{i=1}^{d-1} dx_i^2\right], \tag{3.114}$$

where H_A satisfy (3.108) and $\partial_v f = 0$, and γ_A are defined by

$$\gamma_A = \begin{cases} D-2 & y_\alpha \text{ belonging to } q_A\text{-brane} \\ 0 & \text{otherwise} \end{cases}\,. \tag{3.115}$$

The n_A-form field strength is given by

$$F_{\mathbf{n_A}} = -\partial_v H_A^{-1}(u,v) du\wedge dv \wedge dy_2 \wedge \ldots \wedge dy_{q_A+1}\,, \tag{3.116}$$

and the dilaton field is given by

$$e^{\varphi} = \prod_A H_A^{\frac{\epsilon_A a_A}{2}}\,. \tag{3.117}$$

3.2.2 N2 and N5-brane solutions in M-theory

In the previous subsection, we provide the condition for the solutions to be supersymmetric. Since we are interested in Supersymmetry and the BPS solutions, we construct the concrete BPS solutions by introducing the branes which are consistent with the given conditions. Here, since low energy effective M-theory is described by the 11-dimensional Supergravity, and some of the solutions in the 10-dimensional Supergravity can be related with the 11-dimensional ones, we discuss BPS pp-wave brane solutions in the context of the 11-dimensional Supergravity here.

In the 11-dimensional Supergravity, there is only 3-form field, thus there is no dilaton φ. Setting $D = 11$ and $a_A = 0$. For the 3-form field, because $n_A = 4$, the electric type field is related to N2-brane. Therefore, the solutions with one electrically charged N-brane are then written as

$$
\begin{aligned}
ds_{11}^2 &= H_2^{1/3}\left[H_2^{-1}2du(dv+fdu)+H_2^{-1}(dy_2^2+dy_3^2)+\sum_{i=1}^{7}dx_i^2\right],\\
F_4 &= dH_2^{-1}du\wedge dy_2\wedge dy_6\,,
\end{aligned}
\tag{3.118}
$$

where H_2 is a harmonic function depending on v.

On the other hand, the magnetic type field is related to N5-brane because $\tilde{q}_A = \tilde{n}_A - 2 = D - n_A - 2 = 5$. Therefore, the solutions with one magnetically charged N-brane are given by

$$
\begin{aligned}
ds_{11}^2 &= H_5^{2/3}\left[H_5^{-1}2du(dv+fdu)+H_5^{-1}\sum_{\alpha=2}^{6}dy_\alpha^2+\sum_{i=1}^{4}dx_i^2\right],\\
*F_4 &= dH_5^{-1}du\wedge dy_2\wedge\ldots\wedge dy_5\,,
\end{aligned}
\tag{3.119}
$$

where H_5 is a harmonic function depending on v.

Of course, it is also possible to introduce the combinations of N2-branes and N5-branes. In such cases, the crossing rule obtained in Eq.(3.106) plays very important role. From the crossing rule, all the possible cases for the intersecting dimensions are,

$$
M2\cap M2\to\bar{q}=0\,,\quad M2\cap M5\to\bar{q}=1\,,\quad M5\cap M5\to\bar{q}=3\,. \tag{3.120}
$$

Among them, we obtain $d = 4$ case with the BPS pp-wave solutions uniquely as follow,

$$
\begin{aligned}
ds_{11}^2 = {}& H_2^{1/3}H_5^{2/3}[(H_2H_5)^{-1}2du(dv+fdu)+(H_2H_5)^{-1}dy_2^2\\
&+H_5^{-1}(dy_3^2+\cdots+dy_6^2)+H_2^{-1}dy_7^2+dx_1^2+dx_2^2+dx_3^2]\,,
\end{aligned}
\tag{3.121}
$$

in which N2-brane occupies the u, y^2 and y^7 directions and N5-brane occupies the $u, y^2 \ldots, y^6$ directions.

It is important to discuss the possibility to generate other supersymmetric solutions based on the solutions obtained above. In general, the dimensional reduction of 11-dimensional Supergravity provides 10-dimensional theory with two supersymmetries, that is, type IIA Supergravity. In this theory, the gravitinos have opposite chiralities (γ eugenvalues), i.e., it is 'nonchiral'.

There is the other 10-dimensional theory with two supersymmetries which cannot be obtained by the reduction or the truncation of the 11-dimensional theory, that is, type IIB Supergravity. In this theory, the gravitinos have the same chirality, i.e., it is 'chiral'. These describe the leading low energy behaviors of type IIA and type IIB suberstring theory respectively. This fact makes the explicit formulations of the Supergravity theories of particularly interesting.

If we compactify the y^7 coordinate in the solutions above, we obtain the pp-wave solutions in type IIA Supergravity theory in which the dilaton φ with coupling constant $\epsilon a = (3-q)/2$ appears. On the other hand, if we compactify the y^1 coordinate, S-brane solutions in type II Supergravity theories discussed in [76][75] can be obtained. Since for the both of the cases, the harmonic rule is satisfied naturally, we can make all the possible solutions by making use of the T- and S-dual transformations as in BPS Dp-barne cases.

3.2.3 Supersymmetry in N2⊥N5 Solution

Here we discuss Supersymmetry in the solution obtained in null brane. The invariance for Supersymmetry transformation of a gravitino gives a criterion for existence of unbroken Supersymmetry. This condition is given by the Killing equation for the Killing spinor ϵ [53], i.e.,

$$\delta\psi_{\hat{a}} = \left[e_{\hat{a}}{}^{\mu}\partial_{\mu} + \frac{1}{4}w^{\hat{b}\hat{c}}{}_{\hat{a}}\gamma_{\hat{b}\hat{c}} + \frac{1}{288}(\gamma_{\hat{a}}{}^{\hat{b}\hat{c}\hat{d}\hat{e}} - 8\delta_{\hat{a}}^{\hat{b}}\gamma^{\hat{c}\hat{d}\hat{e}})F_{\hat{b}\hat{c}\hat{d}\hat{e}}\right]\epsilon = 0\,, \tag{3.122}$$

where $\gamma_{\hat{a}\hat{b}}$'s are the antisymmetrized products of eleven-dimensional gamma matrices with unit strength on vielbein $e^{\hat{a}}{}_{\mu}$, and a spin connection is given by

$$w^{\hat{b}\hat{c}}{}_{\hat{a}} = e_{\hat{a}}{}^{\mu}\left(e^{\hat{b}\nu}\partial_{[\mu}e^{\hat{c}}{}_{\nu]} - e^{\hat{c}\nu}\partial_{[\mu}e^{\hat{b}}{}_{\nu]} - e^{\hat{b}\rho}e^{\hat{c}\sigma}e_{\hat{d}\mu}\partial_{[\rho}e^{\hat{d}}{}_{\sigma]}\right)\,. \tag{3.123}$$

Now we consider the M2⊥M5 black brane solution related to the five-dimensional black hole, hence the metric is given by

$$ds^2 = H_2^{1/3}H_5^{2/3}\left[2(H_2H_5)^{-1}du\,(dv+f du) + \sum_{\alpha=2}^{5}H_5^{-1}dy_\alpha^2 + H_2^{-1}dy_6^2 + \sum_{i=1}^{3}dx_i^2\right]. \tag{3.124}$$

The non-trivial components of field strength $F_{\hat{a}\hat{b}\hat{c}\hat{d}}$ are given by

$$F_{\hat{u}\hat{v}\hat{y}_2\hat{y}_6} = -H_2^{-1/6}H_5^{-1/3}\frac{\partial_v H_2}{H_2} \tag{3.125}$$

$$F_{\hat{i}\hat{j}\hat{k}\hat{y}_2} = \epsilon^{ijk}H_2^{-1/6}H_5^{-1/3}\frac{\partial_v H_5}{H_5}\,. \tag{3.126}$$

Using Eq. (3.107) and the above explicit expression for $F_{\hat{a}\hat{b}\hat{c}\hat{d}}$, we obtain the Killing equations (3.122) as

$$\begin{aligned}\delta\psi_{\hat{u}} &= H_2^{1/3}H_5^{1/6}(\partial_u - f\partial_v)\epsilon \\ &\quad -\frac{1}{6}H_2^{1/3}H_5^{1/6}\left[\frac{\partial_v H_2}{H_2}\gamma^{\hat{v}\hat{\alpha}_2\hat{\alpha}_7} + \frac{\partial_v H_5}{2H_5}\gamma^{\hat{v}\hat{\alpha}_2\cdots\hat{\alpha}_6}\right]\epsilon\,,\end{aligned} \tag{3.127}$$

$$\delta\psi_{\hat{v}} = H_2^{1/3}H_5^{1/6}\partial_v\epsilon + \frac{1}{6}H_2^{1/3}H_5^{1/6}\left[\frac{\partial_v H_2}{H_2}\gamma^{\hat{u}\hat{\alpha}_2\hat{\alpha}_7} + \frac{\partial_v H_5}{2H_5}\gamma^{\hat{u}\hat{\alpha}_2\cdots\hat{\alpha}_6}\right]\epsilon\,, \quad (3.128)$$

$$\begin{aligned}\delta\psi_{\hat{i}} &= \frac{1}{6}H_2^{1/3}H_5^{1/6}\left[\frac{\partial_v H_2}{H_2}\gamma^{\hat{u}\hat{i}}(1-\gamma^{\hat{u}\hat{\alpha}_2\hat{\alpha}_7}) + \frac{\partial_v H_5}{2H_5}\gamma^{\hat{u}\hat{i}}(1-\gamma^{\hat{u}\hat{\alpha}_2\cdots\hat{\alpha}_6})\right]\epsilon \\ &\quad + \frac{1}{6}H_2^{1/3}H_5^{1/6}\left[\frac{\partial_v H_2}{H_2}\gamma^{\hat{v}\hat{\alpha}_2\hat{\alpha}_7} + \frac{\partial_v H_5}{2H_5}\gamma^{\hat{v}\hat{\alpha}_2\cdots\hat{\alpha}_6}\right]\epsilon\,, \end{aligned} \quad (3.129)$$

$$\begin{aligned}\delta\psi_{\hat{\alpha}_{N5}} &= \frac{1}{6}H_2^{1/3}H_5^{1/6}\left[\frac{\partial_v H_2}{H_2}\gamma^{\hat{u}\hat{\alpha}_{N5}}(1-\gamma^{\hat{u}\hat{\alpha}_2\hat{\alpha}_7}) + \frac{\partial_v H_5}{2H_5}\gamma^{\hat{u}\hat{\alpha}_{N5}}(1-\gamma^{\hat{u}\hat{\alpha}_2\cdots\hat{\alpha}_6})\right]\epsilon \\ &\quad + \frac{1}{6}H_2^{1/3}H_5^{1/6}\left[\frac{\partial_v H_2}{H_2}\gamma^{\hat{v}\hat{\alpha}_2\hat{\alpha}_7} + \frac{\partial_v H_5}{2H_5}\gamma^{\hat{v}\hat{\alpha}_2\cdots\hat{\alpha}_6}\right]\epsilon\,, \end{aligned} \quad (3.130)$$

$$\begin{aligned}\delta\psi_{\hat{\alpha}_7} &= \frac{1}{6}H_2^{-1/6}H_5^{-1/3}\left[\frac{\partial_v H_2}{H_2}\gamma^{\hat{u}\hat{\alpha}_7}(1-\gamma^{\hat{u}\hat{\alpha}_2\hat{\alpha}_7}) + \frac{\partial_v H_5}{2H_5}\gamma^{\hat{u}\hat{\alpha}_7}(1-\gamma^{\hat{u}\hat{\alpha}_2\cdots\hat{\alpha}_6})\right]\epsilon \\ &\quad + \frac{1}{6}H_2^{-1/6}H_5^{-1/3}\left[\frac{\partial_v H_2}{H_2}\gamma^{\hat{v}\hat{\alpha}_2\hat{\alpha}_7} + \frac{\partial_v H_5}{2H_5}\gamma^{\hat{v}\hat{\alpha}_2\cdots\hat{\alpha}_6}\right]\epsilon\,. \end{aligned} \quad (3.131)$$

Most parts of the above equations vanish if we impose the following condition for the Killing spinor ϵ:

$$(1-\gamma^{\hat{u}\hat{\alpha}_2\hat{\alpha}_7})\epsilon = 0\,, \quad (1-\gamma^{\hat{u}\hat{\alpha}_2\cdots\hat{\alpha}_5})\epsilon = 0\,, \quad \gamma^{\hat{v}}\epsilon = 0\,. \quad (3.132)$$

The two terms of the Killing spinor equations, however, are not vanishing, which are

$$(\partial_u - f\partial_v)\epsilon = 0\,, \quad \partial_v\epsilon = -\frac{1}{6}\left(\frac{\partial_v H_2}{H_2} + \frac{\partial_v H_5}{2H_5}\right)\epsilon\,. \quad (3.133)$$

Therefore we can choose the Majonara spinor ϵ as

$$\begin{aligned}&\epsilon = H_2^{-1/6}H_5^{-1/12}\epsilon_0(u) \\ &\partial_u\epsilon_0(u) = \frac{1}{6}\left[\frac{\partial_u H_2 - f\partial_v H_2}{H_2} + \frac{\partial_u H_5 - f\partial_v H_5}{2H_5}\right]\epsilon_0(u)\,. \end{aligned} \quad (3.134)$$

Thus in this case we remain the $1/8$ Supersymmetry in null brane solutions, which arrive at an $N=1$ Supersymmetry in four dimension with torus compactification, we show in below.

Chapter 4

Black Hole Solutions

The black hole solutions in a supergravity include the higher-order effects of a string coupling constant, although these are solutions in a low energy limit. On the other hand, the counting of states of corresponding branes is performed at the lowest order of a string coupling. The results of these two calculations need not coincide each other. However, if there is supersymmetry, these should be the same because the numbers of dynamical freedom cannot be different in these BPS representations. Therefore, supersymmetric black hole (or black ring) solutions are often discussed in many literature [79, 80, 22, 81, 82].

The classification of supersymmetric solutions in minimal $\mathcal{N} = 2$ supergravity in four dimensional spacetime was first performed by a time-like or null Killing spinor [83]. Recently, solutions in minimal $\mathcal{N} = 1$ supergravity in five dimensions have been classified into two classes by use of G-structures analysis [84, 85, 86, 87, 88]. The six-dimensional minimal supergravity has also been discussed [89].

However, the fundamental unified theory is constructed in either ten or eleven dimensions. When we discuss the entropy of black holes, we have to show the relation between those supersymmetric black holes and more fundamental black branes either in ten or eleven dimensions, from which we obtain black holes (or rings) via compactification. Thus we may need to construct more generic black brane solutions in the fundamental theory and the black holes by some compactification. M-theory is the best candidate for such a unified theory. Since its low energy limit coincides with the eleven-dimensional supergravity, it provides a natural framework to study black brane or BPS brane solutions.

In this chapter we consider the lower-dimensional black hole solutions by compactification from eleven-dimensional Supergravity related to M-theory. Our space-time is of course four dimension, but the Superstring and M-theory are defined in ten and eleven dimension. Thus we must do the dimenisonal reduction from ten or eleven dimension to lower dimension. The gauge group in eleven- or ten-dimesional Supergravity is given by $U(1)$ symmetry, thus we try to find the way to non-Abelian gauge group (e.g., $SU(3) \times SU(2) \times U(1)$, or $SU(5) \in E_6 \in E_8$) via compactification. Supersymmetry is also need to break down by the compactification if we have too much components of fileds and spinors. However the case we consider the last chapter, the

Supersymmetry is breaking $1/8$ by intersecting intersecting branes, which is no need to break more Supersymmetry via compactificaiton. Our case also provide a non-trivial gauge group from intersecting branes.

4.1 Kaluza-Klein Compactification

In this section we start from a theory including gravity in $d+p$ dimensions and we reduce it on a general p-dimensional torus with non-trivial metric. The main assumption is that all the fields do not depend on the p internal coordinates, which means that we will only consider the zero modes of these reduced fields. This dimensional reduction is called Kaluza-Klein reduction. By using the Kaluza-Klein reduction, we obtain the new fields which indices is in the compactified space p.

Let us start with a theory in $d+p$ dimensions containing only gravity. We separate p internal directions from the metric, then we find

$$\begin{aligned} ds^2 &\equiv G_{MN} dz^M dz^N \\ &= g_{\mu\nu} dx^\mu dx^\nu + h_{ij}(dy^i + A^i_\mu dx^\mu)(dy^j + A^j_\nu dx^\nu)\,, \end{aligned} \tag{4.1}$$

where $z^M = (x^\mu, y^i)$ with $\mu = 0 \ldots d-1$ and $i = 1 \ldots p$, and all the functions appearing in (4.1) depend only on x^μ. The components of the $d+p$-dimensional metric G_{MN} can be expressed as

$$G_{\mu\nu} = g_{\mu\nu} + h_{ij} A^i_\mu A^j_\nu, \quad G_{\mu i} = h_{ij} A^j_\mu, \quad G_{ij} = h_{ij}\,. \tag{4.2}$$

We consider the following action for pure gravity in $d+p$ dimensions as

$$S = \int d^{d+p} z \sqrt{-G}\mathcal{R}[G]\,. \tag{4.3}$$

$\mathcal{R}[G]$ denotes the curvature constructed from the metric G_{MN}. The relation between the determinants is $\sqrt{-G} = \sqrt{-g}\sqrt{h}$. The curvature can also be written in d-dimensional components. Using the equations (4.2) we find

$$\mathcal{R}[G] = \mathcal{R}[g] - \frac{3}{4}(\partial_\mu h^{ij})^2 - \frac{1}{4}(h^{ij}\partial_\mu h_{ij})^2 - h^{ij}\Box h_{ij} - \frac{1}{4} h_{ij} F^i_{\mu\nu} F^{j\mu\nu}\,, \tag{4.4}$$

where $\mathcal{R}[g]$ and $\Box$ are the d-dimensional curvature and Dalembertian operator from the metric $g_{\mu\nu}$. The field strength is given by $F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu$ and all the greek indices are raised with $g^{\mu\nu}$. We can get rid of the Dalembertian form the curvature by a total divergence of $\sqrt{h}$ as

$$\frac{1}{\sqrt{h}}\Box\sqrt{h} = \frac{1}{2} h^{ij}\Box h_{ij} + \frac{1}{2}(\partial_\mu h^{ij})^2 + \frac{1}{4}(h^{ij}\partial_\mu h_{ij})^2\,. \tag{4.5}$$

Thus the action (4.3) is equivalent to

$$S = \int d^d x \sqrt{-g}\sqrt{h}\left[\mathcal{R}[g] + \frac{1}{4}(\partial_\mu h^{ij})^2 + \frac{1}{4}(h^{ij}\partial_\mu h_{ij})^2 - \frac{1}{4} h_{ij} F^i_{\mu\nu} F^{j\mu\nu}\right]\,. \tag{4.6}$$

In order to put this action into its canonical form in Einstein frame, we now want to eliminate the factor of $\sqrt{h}$ in front of $R[g]$. We have to operate a conformal transformation of the metric. Most generally transformation of the metric is given by

$$g_{\mu\nu} = e^{2\varphi}\tilde{g}_{\mu\nu}\,, \tag{4.7}$$

and the curvature in d dimensions transforms as

$$\mathcal{R}[g] = e^{-2\varphi}\left[\mathcal{R}[\tilde{g}] - 2(d-1)\Box\varphi - (d-1)(d-2)\partial_\mu\varphi\partial^\mu\varphi\right]\,, \tag{4.8}$$

where the metric entering in the quantities on the right hand side is always $\tilde{g}_{\mu\nu}$.

The conformal transformation to the Einstein frame is given by

$$g_{\mu\nu} = e^{-\frac{1}{d-2}\log h}\tilde{g}_{\mu\nu}\,. \tag{4.9}$$

We can again drop a total divergence arising from the term proportional to $\Box\log h$, then we finally obtain the Kaluza-Klein reduced action in d dimension in canonical form as

$$S = \int d^dx\sqrt{-g}\left[\mathcal{R} + \frac{1}{4}(\partial_\mu h^{ij})^2 - \frac{1}{4(d-2)}(h^{ij}\partial_\mu h_{ij})^2 - \frac{1}{4}h^{\frac{1}{d-2}}h_{ij}F^i_{\mu\nu}F^{j\mu\nu}\right]\,. \tag{4.10}$$

Now we specialize to the simplest case in which we reduce on a single direction y, and we can write $h_{yy} \equiv h = e^{2\sigma}$. The action (4.10) for this case becomes

$$S = \int d^dx\sqrt{-g}\left[\mathcal{R} - \frac{d-1}{d-2}(\partial_\mu\sigma)^2 - \frac{1}{4}e^{2\frac{d-1}{d-2}\sigma}F_{\mu\nu}F^{\mu\nu}\right]\,. \tag{4.11}$$

The kinetic term of the scalar field σ can be put to its canonical form simply defining a new scalar $\phi = a\sigma$ such that its kinetic term is multiplied by $1/2$. This requirement gives for $a = \sqrt{2(d-1)/d-2}$, and the action becomes simply as

$$S = \int d^dx\sqrt{-g}\left[\mathcal{R} - \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{4}e^{a\phi}F_{\mu\nu}F^{\mu\nu}\right]\,. \tag{4.12}$$

First we consider the four-dimensional physics is seen as a reduction from 5 dimensions with $a = \sqrt{3}$. It is this class of 4 dimensional dilatonic black holes that will be relevant to the study of Kaluza-Klein monopoles. Next the reduction from eleven-dimensional Supergravity to ten-dimensional ones with $a = 3/2$. We understand the physical meaning of scalar field $\langle e^\phi\rangle = g_s$ as a ten-dimensional string coupling constant, and $\langle e^\sigma\rangle \sim R_{11}/l_p$ as the size of the eleventh direction in eleven-dimensional Planck units. Using the relations between $\phi = 3\sigma/2$, we find $R_{11} \sim g_s^{2/3}l_p$, which means that the compactified scales are related to the string coupling.

Now we consider with the matter fields, which is in particular the anti-symmetric tensor fields, reduce under Kaluza-Klein compactification. For simplicity, we choose also in the simplest case of the reduction on a single direction. Suppose in $d+1$ dimensions we have a n-form field strength deriving from a potential

$$\begin{aligned} H_{M_1\ldots M_n} &= n\partial_{[M_1}C_{M_2\ldots M_n]} \\ &= \partial_{M_1}C_{M_2\ldots M_n} + (\text{cyclic permutations})\,. \end{aligned} \tag{4.13}$$

Upon reduction on y, the $(n-1)$-form potential C gives rise to two potentials A^{n-1} and A^{n-2} which are an $(n-1)$- and an $(n-2)$-form as

$$A^{n-1}_{\mu_1\dots\mu_{n-1}} = C_{\mu_1\dots\mu_{n-1}}, \quad A^{n-2}_{\mu_1\dots\mu_{n-2}} = C_{\mu_1\dots\mu_{n-2}y} \,. \tag{4.14}$$

The corresponding field strengths are found to be

$$F^{n}_{\mu_1\dots\mu_n} = n\partial_{[\mu_1}A^{n-1}_{\mu_2\dots\mu_n]} = H_{\mu_1\dots\mu_n} \tag{4.15}$$
$$F^{n-1}_{\mu_1\dots\mu_{n-1}} = (n-1)\partial_{[\mu_1}A^{n-2}_{\mu_2\dots\mu_{n-1}]} = H_{\mu_1\dots\mu_{n-1}y} \,. \tag{4.16}$$

The Lagrangian in $d+1$ dimensions of the anti-symmetric tensor is given by

$$L = -\frac{1}{2n!}H_{M_1\dots M_n}H^{M_1\dots M_n} \,, \tag{4.17}$$

and after dimensional reduction we find

$$L = -\frac{1}{2(n-1)!}e^{-2\sigma}F^{n-1}_{\mu_1\dots\mu_{n-1}}F^{(n-1)\mu_1\dots\mu_{n-1}} - \frac{1}{2n!}F'^{n}_{\mu_1\dots\mu_n}F'^{(n)\mu_1\dots\mu_n} \,, \tag{4.18}$$

where we have defined the modified n-form field strength as

$$F'^{n}_{\mu_1\dots\mu_n} = F^{n}_{\mu_1\dots\mu_n} - nF^{n-1}_{[\mu_1\dots\mu_{n-1}}A_{\mu_n]} \,. \tag{4.19}$$

The Kaluza-Klein procedure introduces this Chern-Simons-like coupling.

The action for the n-form can be rewriteen in the Einstein frame like in (4.12) as

$$\begin{aligned} S &= \int d^{d+1}z\sqrt{-G}\left[-\frac{1}{2n!}H^2\right] \\ &= \int d^dx\sqrt{-g}\left[-\frac{1}{2(n-1)!}e^{a_{n-1}\phi}F^2_{(n-1)} - \frac{1}{2n!}e^{a_n\phi}F'^2_{(n)}\right] \,, \end{aligned} \tag{4.20}$$

where the couplings to the scalar field are given by

$$a_{n-1} = -(d-n)\sqrt{\frac{2}{(d-1)(d-2)}} \,, \quad a_n = (n-1)\sqrt{\frac{2}{(d-1)(d-2)}} \,. \tag{4.21}$$

The reduction of the 4-form appearing in eleven-dimensional Supergravity, we obtain for the 3- and 4-form of the ten-dimensional type IIA Supergravity respectively $a_3 = -1$ and $a_4 = 1/2$.

Now we consider our black brane solutions obtained in last chapter, and after the Kaluza-Klein compactification, we will find the black hole solutions represented by intersecting M-branes. Rewriting the following part of the metric (3.53) as

$$\begin{aligned} &2du\left(dv + fdu + \frac{\mathcal{A}}{\sqrt{2}}\right) \\ &= (1+f)\left[dy_1 - \frac{1}{1+f}\left(fdt - \frac{\mathcal{A}}{2}\right)\right]^2 - \frac{1}{1+f}\left(dt + \frac{\mathcal{A}}{2}\right)^2 \,, \end{aligned} \tag{4.22}$$

we obtain our metric in D-dimensions as

$$\begin{aligned} ds_D^2 &= \prod_A H_A^{2\frac{q_A+1}{\Delta_A}} \left[-\prod_B H_B^{-2\frac{D-2}{\Delta_B}} \frac{1}{1+f} \left(dt + \frac{\mathcal{A}}{2} \right)^2 + \sum_{i=1}^{d-1} dx_i^2 \right] \\ &+ \prod_A H_A^{-2\frac{D-q_A-3}{\Delta_A}} (1+f) \left[dy_1 - \frac{f dt - \mathcal{A}/2}{1+f} \right]^2 + \sum_{\alpha=2}^{p} \prod_A H_A^{-2\frac{\delta_{\alpha A}}{\Delta_A}} dy_\alpha^2 \,. \end{aligned} \tag{4.23}$$

Introducing the conformal factors Ω_1, Ω_α and Ω by

$$\begin{aligned} \Omega_1^2 &= (1+f) \prod_A H_A^{-2\frac{D-q_A-3}{\Delta_A}} \\ \Omega_\alpha^2 &= \prod_A H_A^{-2\frac{\delta_{\alpha A}}{\Delta_A}} \qquad (\alpha = 2, \cdots, p) \\ \Omega^2 &= \prod_{\alpha=1}^{p} \Omega_\alpha^2 = (1+f) \prod_A H_A^{\frac{2}{\Delta_A}[D-d-q_A(d-2)]} \,, \end{aligned} \tag{4.24}$$

we perform a conformal transformation of our metric as

$$ds_D^2 = \Omega^{-\frac{2}{d-2}} d\bar{s}_d^2 + \Omega_1^2 \left[dy_1 - \frac{1}{1+f} \left(f dt - \frac{\mathcal{A}}{2} \right) \right]^2 + \sum_{\alpha=2}^{p} \Omega_\alpha^2 dy_\alpha^2 \,. \tag{4.25}$$

With this conformal transformation, we obtain the Einstein gravity in d-dimensions, which metric is given by

$$\begin{aligned} d\bar{s}_d^2 &\equiv \bar{g}_{\bar{\mu}\bar{\nu}} dx^{\bar{\mu}} dx^{\bar{\nu}} \\ &= -\Xi^{d-3} \left(dt + \frac{\mathcal{A}}{2} \right)^2 + \Xi^{-1} \sum_{i=1}^{d-1} dx_i^2 , \\ \Xi &\equiv (1+f)^{-1/(d-2)} \prod_A H_A^{-\frac{2(D-2)}{(d-2)\Delta_A}} \,, \end{aligned} \tag{4.26}$$

where $\bar{\mu}, \bar{\nu}, \cdots$ are coordinate indices for d-dimensional spacetime. If the compactified space is sufficiently small, we find the effective d-dimensional world with the metric ((4.26)).

If this spacetime is asymptotically flat, which we impose, it may describe a black object in d-dimensions. From the asymptotic form of the metric, we can define the ADM mass $M_{\rm ADM}$ as

$$\bar{g}_{00} \sim -1 + \frac{16\pi G_d}{(d-2)\omega_{d-2}} \frac{M_{\rm ADM}}{r^{d-3}} \,, \tag{4.27}$$

where

$$\omega_{d-2} \equiv \frac{2\pi^{\frac{d-1}{2}}}{\Gamma\left(\frac{d-1}{2}\right)}, \quad \text{and} \quad r^2 \equiv \sum_{i=1}^{d-1} x_i^2 \,. \tag{4.28}$$

Assuming

$$\begin{aligned} H_A &\to 1+\frac{\mathcal{Q}_H^{(A)}}{r^{d-3}} \\ f &\to \frac{\mathcal{Q}_0}{r^{d-3}}\,, \end{aligned} \tag{4.29}$$

we obtain

$$M_{\mathrm{ADM}}=\frac{(d-3)\pi^{\frac{d-3}{2}}}{8G_d\Gamma\left(\frac{d-1}{2}\right)}\left[\mathcal{Q}_0+\sum_A \frac{2(D-2)}{\Delta_A}\mathcal{Q}_H^{(A)}\right]. \tag{4.30}$$

For the case of eleven-dimensional M theory (and ten-dimensional type IIB string theory), we find

$$\begin{aligned} \Xi &= \left[(1+f)\prod_A H_A\right]^{-1/(d-2)} \\ M_{\mathrm{ADM}} &= \frac{(d-3)\pi^{\frac{d-3}{2}}}{8G_d\Gamma\left(\frac{d-1}{2}\right)}\left[\mathcal{Q}_0+\sum_{A'}\mathcal{Q}_H^{(A')}\right], \end{aligned} \tag{4.31}$$

where A' denotes charged branes.

Once we find solutions described by the above set of equations, we have to study a spacetime structure. In particular, the horizon and the singularity of a spacetime are important geometrical objects. We then have to evaluate the curvature invariant of the metric ((4.26)). We calculate the Kretschmann invariant, which is given by

$$\begin{aligned} \bar{\mathcal{R}}_{\mu\nu\rho\sigma}\bar{\mathcal{R}}^{\mu\nu\rho\sigma} &= \frac{1}{128}\Xi^{2(d-1)}\left[3\mathcal{F}_{ij}^4+5(\mathcal{F}_{ik}\mathcal{F}_j{}^k)^2\right]-\frac{1}{16}\Xi^d\left[4(\partial_i\mathcal{F}_{kl})^2+6(d-2)\partial^i X\,\partial_i(\mathcal{F}_{kl}^2)\right. \\ &\quad -4d\partial^i\partial^j X\mathcal{F}_{ik}\mathcal{F}_j{}^k+2(3d^2-18d+22)(\partial^i X\mathcal{F}_{ij})^2+\left.(4d-9)(\partial X)^2\mathcal{F}_{ij}^2\right] \\ &\quad +\frac{1}{8}\Xi^2\left[8(d-2)(d-3)(\partial_i\partial_j X)^2+8(\partial^2 X)^2+8(d-2)^2(d-3)\partial^i X\partial^j X\,\partial_i\partial_j X\right. \\ &\quad -8(d-2)(d-3)\partial^2 X(\partial X)^2+\left.(d-2)(d-3)(2d^2-8d+7)(\partial X)^4\right], \end{aligned} \tag{4.32}$$

where $X\equiv\ln\Xi$.In what follows, we present the exact solutions for $D=11$ and $d=5$. For $D=10$, the construction of solutions is almost the same as that of $D=11$.

We will also possible to compactify the extra spacetime using the infinite periodic sequence of black hole solutions. In this case the metric is the same as one of four-dimenisonal asymptotically, but near horizon we will find extra-dimensioinal aspect [94].

4.2 Five-dimensional Black Hole Solutions

We consider solutions in five-dimensions. There are two branes (M2 and M5). Then $N_{A'}+N_{A''}=2$. In the ten-dimensional type IIB case, we find the exactly the same as what we show below, when we replace M2 and M5 with D1 and D5 (the indices $A = 2, 5$ with the indices $A = 1, 5$).

The metric in five-dimensions is written by

$$d\bar{s}_5^2 = -\Xi^2 \left(dt + \frac{\mathcal{A}}{2} \right)^2 + \Xi^{-1} ds_{\mathbb{E}^4}^2 \,, \tag{4.33}$$

where $\Xi = [H_2 H_5 (1+f)]^{-1/3}$. The unknown functions $H_A (A = 2, 5)$, $\mathcal{A}_i$ and f satisfy the following equations:

$$\partial^2 H_A = 0 \quad (A = 2, 5) \tag{4.34}$$

$$\partial_j \mathcal{F}^{ij} = 0 \tag{4.35}$$

$$\partial^2 f = \mathcal{S} \equiv \frac{\beta}{8 H_2 H_5} \mathcal{F}^{ij} \mathcal{F}_{ij} \,, \tag{4.36}$$

where

$$\mathcal{F}_{ij} = \partial_i \mathcal{A}_j - \partial_j \mathcal{A}_i$$

$$\beta = 1 - \frac{1}{2} \left(N_{A'} + \sum_{A''} \lambda_{A''}^2 \right) \,, \tag{4.37}$$

which value is explicitly given in Table 4.1.

	M_2	M_5	type of source branes	β	H_A
(1)	$\mathcal{C}$	$\mathcal{C}$	two charged branes	0	H_2, H_5 : h.f.
(2a)	$\mathcal{C}$	$\mathcal{N}$	charged & neutral branes	$\frac{1}{2}\left(1-\lambda_5^2\right)$	H_2 : h.f., $H_5 = 1$
(2b)	$\mathcal{N}$	$\mathcal{C}$	neutral & charged branes	$\frac{1}{2}\left(1-\lambda_2^2\right)$	$H_2 = 1$, H_5: h.f.
(2c)	$\mathcal{N}$	$\mathcal{N}$	two neutral branes	$1-\frac{1}{2}\left(\lambda_2^2+\lambda_5^2\right)$	$H_2 = H_5 = 1$

Table 4.1: The type of source branes and the value of β. There are two branes (M2 and M5). $\mathcal{C}$ and $\mathcal{N}$ denote a charged brane and a neutral brane with a current. λ_2 and λ_5 are arbitrary parameters, which correspond to current strength. h.f. denote the harmonic function of d dimensional space coordinate.

In order to find the exact solutions, we assume that the 4-dimensional x-space has two rotation symmetries which Killing vectors ($\xi^i_{(\phi)}$ and $\xi^i_{(\psi)}$) commute each other. In this case, Eq. (4.35) is reduced to two uncoupled equations for two scalar fields, $\mathcal{A}_\phi = \mathcal{A}_i \xi^i_{(\phi)}$ and

$\mathcal{A}_\psi = \mathcal{A}_i \xi^i_{(\psi)}$, as

$$\partial^2 \mathcal{A}_\phi - \partial_i \ln \left(\xi_{(\phi)} \cdot \xi_{(\phi)}\right) \, \partial^i \mathcal{A}_\phi = 0 \,, \tag{4.38}$$

$$\partial^2 \mathcal{A}_\psi - \partial_i \ln \left(\xi_{(\psi)} \cdot \xi_{(\psi)}\right) \, \partial^i \mathcal{A}_\psi = 0 \,. \tag{4.39}$$

Here we have assumed that the other components of $\mathcal{A}_i$ vanish.

We now have the Laplace equations or similar equations (the Poisson equation or Eqs. (4.38) and (4.39)) for several scalar functions (H_A, $\mathcal{A}_\phi$, $\mathcal{A}_\psi$, and f). Each equation is linear and uncoupled except for the equation for f with $\beta \neq 0$. Hence it is very easy to find general solutions because the Laplace-Beltrami operator is defined on the flat Euclidian space. Once we obtain a complete set of solutions in an appropriate curvilinear coordinate system, we can construct any solutions by superposing them.

We start with the metric form of the four-dimensional Euclidian space written by some orthonormal curvilinear coordinates, i.e.,

$$ds^2_{\mathbb{E}^4} = h_{\xi\xi} d\xi^2 + h_{\eta\eta} d\eta^2 + h_{\phi\phi} d\phi^2 + h_{\psi\psi} d\psi^2 \,. \tag{4.40}$$

We assume that there are two rotation symmetries, as discussed in the text (§IV). Hence h_{ij} depends only on two coordinates: ξ and η.

Eqs. (4.34), (4.38), (4.39) and (4.36) are explicitly written as

$$\partial_\xi \left(\sqrt{\frac{h_{\eta\eta} h_{\phi\phi} h_{\psi\psi}}{h_{\xi\xi}}} \partial_\xi H_A \right) + \partial_\eta \left(\sqrt{\frac{h_{\xi\xi} h_{\phi\phi} h_{\psi\psi}}{h_{\eta\eta}}} \partial_\eta H_A \right) = 0 \quad (A = 2, 5) \,, \tag{4.41}$$

$$\partial_\xi \left(\sqrt{\frac{h_{\eta\eta} h_{\psi\psi}}{h_{\xi\xi} h_{\phi\phi}}} \partial_\xi \mathcal{A}_\phi \right) + \partial_\eta \left(\sqrt{\frac{h_{\xi\xi} h_{\psi\psi}}{h_{\eta\eta} h_{\phi\phi}}} \partial_\eta \mathcal{A}_\phi \right) = 0 \tag{4.42}$$

$$\partial_\xi \left(\sqrt{\frac{h_{\eta\eta} h_{\phi\phi}}{h_{\xi\xi} h_{\psi\psi}}} \partial_\xi \mathcal{A}_\psi \right) + \partial_\eta \left(\sqrt{\frac{h_{\xi\xi} h_{\phi\phi}}{h_{\eta\eta} h_{\psi\psi}}} \partial_\eta \mathcal{A}_\psi \right) = 0 \,, \tag{4.43}$$

$$\begin{aligned} &\partial_\xi \left(\sqrt{\frac{h_{\eta\eta} h_{\phi\phi} h_{\psi\psi}}{h_{\xi\xi}}} \partial_\xi f \right) + \partial_\eta \left(\sqrt{\frac{h_{\xi\xi} h_{\phi\phi} h_{\psi\psi}}{h_{\eta\eta}}} \partial_\eta f \right) \\ &= \frac{\beta}{4 H_2 H_5} \left[\sqrt{\frac{h_{\eta\eta} h_{\psi\psi}}{h_{\xi\xi} h_{\phi\phi}}} (\partial_\xi \mathcal{A}_\phi)^2 + \sqrt{\frac{h_{\xi\xi} h_{\psi\psi}}{h_{\eta\eta} h_{\phi\phi}}} (\partial_\eta \mathcal{A}_\phi)^2 \right. \\ &\qquad \left. + \sqrt{\frac{h_{\eta\eta} h_{\phi\phi}}{h_{\xi\xi} h_{\psi\psi}}} (\partial_\xi \mathcal{A}_\psi)^2 + \sqrt{\frac{h_{\xi\xi} h_{\phi\phi}}{h_{\eta\eta} h_{\psi\psi}}} (\partial_\eta \mathcal{A}_\psi)^2 \right] \,. \end{aligned} \tag{4.44}$$

Giving an explicit form of a solution, we obtain the properties of a black object. For example, assuming the asymptotic behaviors for H_A and f as $H_A \to 1 + \mathcal{Q}_H^{(A)}/r^2$ and $f \to \mathcal{Q}_0/r^2$ as $r \equiv \sqrt{x_1^2 + \cdots + x_4^2} \to \infty$. Then we find Arnowitt-Deser-Misner mass as

$$M_{\rm ADM} = \frac{\pi}{4G_5}\left(\mathcal{Q}_0 + \mathcal{Q}_H^{(2)} + \mathcal{Q}_H^{(5)}\right) . \tag{4.45}$$

The entropy of a black hole, if it exists, is defined by

$$S \;=\; \frac{A_h}{4G_5} , \tag{4.46}$$

where A_h is the area of horizon.

In what follows, adopting the hyperspherical coordinates and the other coordinates as a curvilinear coordinate system, we show explicitly how to construct the exact solutions.

4.3 Hyperspherical Coordinates

We adopt the hyperspherical coordinates:

$$x_1 + ix_2 = r\cos\theta e^{i\phi}, \quad x_3 + ix_4 = -r\sin\theta e^{i\psi} , \tag{4.47}$$

where $0 \le \phi, \psi < 2\pi$ and $0 \le \theta \le \pi/2$. The line element of 4D flat space is

$$ds^2_{\mathbb{E}^4} = dr^2 + r^2\left(d\theta^2 + \cos^2\theta d\phi^2 + \sin^2\theta d\psi^2\right) . \tag{4.48}$$

The symmetric axis is described by $\theta = 0$ and $\pi/2$, and the infinity corresponds to $r = \infty$.

Eq. (4.34) in this coordinate system is

$$\frac{1}{r}\partial_r\left(r^3\partial_r H_A\right) + \frac{1}{\sin\theta\cos\theta}\partial_\theta\left(\sin\theta\cos\theta\partial_\theta H_A\right) = 0 . \tag{4.49}$$

Setting $H_A = h_A(r)j_A(\theta)$, we separate the variables and obtain two ordinary differential equations:

$$\frac{1}{r}\frac{d}{dr}\left(r^3\frac{dh_A}{dr}\right) - Mh_A = 0 , \tag{4.50}$$

$$\frac{1}{\sin\theta\cos\theta}\frac{d}{d\theta}\left(\sin\theta\cos\theta\frac{dj_A}{d\theta}\right) + Mj_A = 0 , \tag{4.51}$$

where M is a separation constant. Eq. (4.51) with $\mu = \cos 2\theta$ is just the Legendre equation as

$$\frac{d}{d\mu}\left((1-\mu^2)\frac{dj_A}{d\mu}\right) + \frac{M}{4}j_A = 0 . \tag{4.52}$$

From regularity conditions on the symmetric axis ($\theta = 0, \pi/2$), we obtain $j_A = P_\ell(\cos 2\theta)$ by setting $M = 4\ell(\ell+1)$ ($\ell = 0, 1, 2, \cdots$). Eq. (4.50) is easily solved as $h_A = r^{2\ell}$ or $r^{-2(\ell+1)}$. The general solution for H_A is then

$$H_A = \sum_{\ell=0}^{\infty} \left[g_\ell^{(A)} r^{2\ell} + h_\ell^{(A)} r^{-2(\ell+1)} \right] P_\ell(\cos 2\theta)\,, \tag{4.53}$$

where $g_\ell^{(A)}$ and $h_\ell^{(A)}$ are arbitrary constants.

From the asymptotically flatness condition, the solution is given by

$$H_A = 1 + \sum_{\ell=0}^{\infty} h_\ell^{(A)} r^{-2(\ell+1)} P_\ell(\cos 2\theta)\,. \tag{4.54}$$

The spherically symmetric solution ($\ell = 0$) is given by

$$H_A = 1 + \frac{\mathcal{Q}_H^{(A)}}{r^2}\,, \tag{4.55}$$

where $\mathcal{Q}_H^{(A)}$ is a constant, which corresponds to a conserved charge.

Next, we discuss Eqs. (4.38) and (4.39), which are written as

$$r\partial_r \left(r\partial_r \mathcal{A}_\phi\right) + \cot\theta\partial_\theta \left(\tan\theta\partial_\theta \mathcal{A}_\phi\right) = 0\,, \tag{4.56}$$
$$r\partial_r \left(r\partial_r \mathcal{A}_\psi\right) + \tan\theta\partial_\theta \left(\cot\theta\partial_\theta \mathcal{A}_\psi\right) = 0\,. \tag{4.57}$$

Setting $\mathcal{A}_\phi = a_\phi(r) b_\phi(\theta)$ and $\mathcal{A}_\psi = a_\psi(r) b_\psi(\theta)$, we have the following ordinary differential equations:

$$r\frac{d}{dr}\left(r\frac{d}{dr}a_\phi\right) - K a_\phi = 0 \tag{4.58}$$
$$\frac{d^2 b_\phi}{d\mu^2} - \frac{1}{1-\mu}\frac{db_\phi}{d\mu} + \frac{K}{4(1-\mu^2)} b_\phi = 0\,, \tag{4.59}$$
$$r\frac{d}{dr}\left(r\frac{d}{dr}a_\psi\right) - L a_\psi = 0 \tag{4.60}$$
$$\frac{d^2 b_\psi}{d\mu^2} + \frac{1}{1+\mu}\frac{db_\psi}{d\mu} + \frac{L}{4(1-\mu^2)} b_\psi = 0\,, \tag{4.61}$$

where $\mu = \cos 2\theta$, and K and L are separation constants. The solutions of Eqs. (4.59) and (4.61) are described by Gauss's hypergeometric functions as $b_\phi(\mu) = F(-\sqrt{K}/2, \sqrt{K}/2, 1, (1-\mu)/2)$ and $b_\psi(\mu) = F(-\sqrt{L}/2, \sqrt{L}/2, 1, (1+\mu)/2)$. The Gauss's hyper geometrical function $F(\alpha, \beta, \gamma, z)$ is defined by

$$F(\alpha,\beta,\gamma,z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n)\Gamma(\beta+n)}{\Gamma(\gamma+n)} \frac{z^n}{n!}\,. \tag{4.62}$$

From regularity conditions, we have to impose that $K = 4m^2$ and $L = 4n^2$, where $m, n = 1, 2, \cdots$. We then have the angular solutions as $b_\phi = F(-m, m, 1, \sin^2\theta)$ and $b_\psi = F(-n, n, 1, \cos^2\theta)$. The explicit forms of this hypergeometric function with $m, n = 1, 2$ are given by

$$\begin{aligned} F(-1,1,1,z) &= 1 - z \\ F(-2,2,1,z) &= 1 - 4z + 3z^2 \,. \end{aligned} \tag{4.63}$$

The equations for a_ϕ and a_ψ are easily solved, i.e., $a_\phi = r^{2m}, r^{-2m}$ and $a_\psi = r^{2n}, r^{-2n}$. We then obtain a general solution for $\mathcal{A}_i$ as

$$\mathcal{A}_\phi = \sum_{m=1}^{\infty} \left[a_m^{(\phi)} r^{2m} + b_m^{(\phi)} r^{-2m}\right] F(-m, m, 1, \sin^2\theta) \tag{4.64}$$

$$\mathcal{A}_\psi = \sum_{n=1}^{\infty} \left[a_n^{(\psi)} r^{2n} + b_n^{(\psi)} r^{-2n}\right] F(-n, n, 1, \cos^2\theta)\,, \tag{4.65}$$

where $a_m^{(\phi)}$, $b_m^{(\phi)}$, $a_n^{(\psi)}$ and $b_n^{(\psi)}$ are arbitrary constants.

Assuming asymptotically flatness, the solution for $\mathcal{A}_i$ is given by

$$\mathcal{A}_\phi = \sum_{m=1}^{\infty} \frac{b_m^{(\phi)}}{r^{2m}} F(-m, m, 1, \sin^2\theta), \tag{4.66}$$

$$\mathcal{A}_\psi = \sum_{n=1}^{\infty} \frac{b_n^{(\psi)}}{r^{2n}} F(-n, n, 1, \cos^2\theta)\,. \tag{4.67}$$

If we take the first two terms in the general solution, we obtain a simple solution as

$$\mathcal{A}_\phi = \frac{\cos^2\theta}{r^2}\left[J_1^{(\phi)} + \frac{J_2^{(\phi)}}{r^2}(1 - 3\sin^2\theta)\right] \tag{4.68}$$

$$\mathcal{A}_\psi = \frac{\sin^2\theta}{r^2}\left[J_1^{(\psi)} + \frac{J_2^{(\psi)}}{r^2}(1 - 3\cos^2\theta)\right]\,, \tag{4.69}$$

where $J_1^{(\phi)}$, $J_1^{(\psi)}$, $J_2^{(\phi)}$ and $J_2^{(\psi)}$ are constants. The first two constants describe angular momenta of a black object. As we show in before, if $\mathcal{F}_{ij}$ is self-dual, the spacetime is supersymmetric. This condition implies $J_1^{(\phi)} = -J_1^{(\psi)}$ and $J_2^{(\phi)} = J_2^{(\psi)}$.

Finally we discuss Eq. (4.36):

$$\begin{aligned} &\frac{1}{r}\partial_r(r^3\partial_r f) + \frac{1}{\sin\theta\cos\theta}\partial_\theta(\sin\theta\cos\theta\partial_\theta f) = \mathcal{S}(r,\theta) \\ &\equiv \frac{\beta}{4H_2H_5}\left[\frac{1}{\cos^2\theta}\left((\partial_r\mathcal{A}_\phi)^2 + \frac{1}{r^2}(\partial_\theta\mathcal{A}_\phi)^2\right) + \frac{1}{\sin^2\theta}\left((\partial_r\mathcal{A}_\psi)^2 + \frac{1}{r^2}(\partial_\theta\mathcal{A}_\psi)^2\right)\right] \end{aligned} \tag{4.70}$$

4.3.1 BMPV Type Solutions: $\beta = 0$

If $\beta = 0$, i.e., Case (1) (two charged brane) or Case (2a-2c) (neutral branes) with appropriately chosen current strength $\lambda_{A''}$, we find the Laplace equation for f, which gives us a simple solution:

$$f = \sum_{\ell=0}^{\infty} \mathcal{Q}_\ell r^{-2(\ell+1)} P_\ell(\cos 2\theta)\,, \tag{4.71}$$

where $\mathcal{Q}_\ell$'s are constants.

In this case, the solution with the lowest multipole moment is given by

$$\begin{aligned} H_A &= 1 + \frac{\mathcal{Q}_H^{(A)}}{r^2} \quad (A = 2, 5)\,, \\ f &= \frac{\mathcal{Q}_0}{r^2}\,, \\ \mathcal{A}_\phi &= \frac{J_\phi \cos^2\theta}{r^2}\,, \\ \mathcal{A}_\psi &= \frac{J_\psi \sin^2\theta}{r^2}\,. \end{aligned} \tag{4.72}$$

The mass and the entropy of this spacetime are

$$M_{\rm ADM} = \frac{\pi}{4G_5}(\mathcal{Q}_0 + \mathcal{Q}_H^{(2)} + \mathcal{Q}_H^{(5)}), \tag{4.73}$$

$$S = \frac{A_h}{4G_5} = \frac{\pi^2}{3G_5} \frac{\Lambda_+^2 + \Lambda_+\Lambda_- + \Lambda_-^2}{\Lambda_+^{3/2} + \Lambda_-^{3/2}}\,, \tag{4.74}$$

where

$$\Lambda_+ = \mathcal{Q}_0 \mathcal{Q}_H^{(2)} \mathcal{Q}_H^{(5)} - \frac{J^2}{8} + \frac{\Delta J^2}{16}\,, \tag{4.75}$$

$$\Lambda_- = \mathcal{Q}_0 \mathcal{Q}_H^{(2)} \mathcal{Q}_H^{(5)} - \frac{J^2}{8} - \frac{\Delta J^2}{16}\,. \tag{4.76}$$

J^2 and ΔJ^2 are defined by $J^2 \equiv (J_\phi^2 + J_\psi^2)/2$ and $\Delta J^2 \equiv J_\phi^2 - J_\psi^2$, respectively.

Fixing J^2, if we maximize entropy S, we find the maximum entropy with

$$S = S_{\rm max} = \frac{\pi^2}{2G_5} \sqrt{\mathcal{Q}_0 \mathcal{Q}_H^{(2)} \mathcal{Q}_H^{(5)} - \frac{J^2}{8}}\,, \tag{4.77}$$

if $\Delta J^2 = 0$, i.e., $J_\phi^2 = J_\psi^2 = J^2$. Note that supersymmetry implies $J_\phi = -J_\psi = J$, which corresponds to the BMPV solution [90, 47]. If $J_\phi \neq -J_\psi$, the above solution describes a regular rotating non-BPS black hole spacetime in five dimensions.

4.3.2 Brinkmann Wave Type Solutions: $\beta \neq 0$

When $\beta \neq 0$, since the source term is quadratic with respect to $\mathcal{A}_i$, it is not so easy to find a general solution. However, once we know the explicit form of the source term, expanding $f(r,\theta)$ and the source term $\mathcal{S}(r,\theta)$ by the Legendre functions as

$$\begin{aligned} f(r,\theta) &= \sum_{\ell=0}^{\infty} f_\ell(r) P_\ell(\cos 2\theta)\,, \\ \mathcal{S}(r,\theta) &= \sum_{\ell=0}^{\infty} \mathcal{S}_\ell(r) P_\ell(\cos 2\theta)\,, \end{aligned} \tag{4.78}$$

we find the ordinary differential equation for each moment ℓ as

$$\frac{1}{r}\frac{d}{dr}\left(r^3 \frac{df_\ell(r)}{dr}\right) - 4\ell(\ell+1) f_\ell(r) = \mathcal{S}_\ell(r)\,. \tag{4.79}$$

If we can integrate this equation, we find an analytic solution.

Here we give one simple example, i.e., $H_2 = H_5 = 1$ [Case (2c) in Table 4.1] with Eqs. (4.68) and (4.69). We find the solutions as

$$f = f_0(r) + f_1(r) P_1(\cos 2\theta) + f_2(r) P_2(\cos 2\theta)\,, \tag{4.80}$$

with

$$\begin{aligned} f_0 &= \frac{\mathcal{Q}_0}{r^2} + \beta\left(\frac{J_1^2}{12r^6} + \frac{J_2^2}{20r^{10}}\right) \\ f_1 &= \frac{\mathcal{Q}_1}{r^4} + \beta\frac{J_{12}^{(\phi\psi)}}{40r^8} \\ f_2 &= \frac{\mathcal{Q}_2}{r^6} + \beta\frac{J_2^2}{14r^{10}}\,, \end{aligned} \tag{4.81}$$

where $\mathcal{Q}_0, \mathcal{Q}_1, \mathcal{Q}_2$ are integration constants and $2J_1^2 = (J_1^{(\phi)})^2 + (J_1^{(\psi)})^2$, $2J_2^2 = (J_2^{(\phi)})^2 + (J_2^{(\psi)})^2$, and $\Delta J_{12}^{(\phi\psi)} = J_1^{(\phi)} J_2^{(\phi)} - J_1^{(\psi)} J_2^{(\psi)}$.

If we set $J_1^{(\phi)} = -J_1^{(\psi)} = J$ and $J_2^{(\phi)} = J_2^{(\psi)} = 0$, we find

$$\begin{aligned} H_2 &= H_5 = 1 \\ f &= \frac{\mathcal{Q}_0}{r^2} + \beta\frac{J^2}{12r^6} \\ \mathcal{A}_\phi &= \frac{J}{r^2}\cos^2\theta\,, \\ \mathcal{A}_\psi &= -\frac{J}{r^2}\sin^2\theta\,. \end{aligned} \tag{4.82}$$

We then recover the Brinkmann solution by setting $\beta = 1$ (i.e., $B_i^A = 0$) [91]. If $\beta < 0$, $(1+f)$ vanishes at $r = r_S (> 0)$, which is a singularity. For the case of $\beta > 0$, this solution is similar to the Brinkmann wave.

We can extend the above solution to Case (2a) in Table 4.1: $H_2 \neq 1$ and $H_5 = 1$ (or (2b):$H_2 = 1$ and $H_5 \neq 1$). We find the following new solution. Supposing that H_2 depends only on r as Eq. (4.55) and the lowest moment for $\mathcal{A}_i$, we can obtain the exact solution:

$$\begin{aligned} H_2 &= 1 + \frac{\mathcal{Q}_H^{(2)}}{r^2} && (4.83) \\ f &= \frac{Q_0}{r^2} + \frac{\beta J^2}{4r^6(H_2-1)^3}\left[H_2^2 - 1 - 2H_2 \ln H_2\right] && (4.84) \\ \mathcal{A}_\phi &= \frac{J_1^{(\phi)}}{r^2}\cos^2\theta && (4.85) \\ \mathcal{A}_\psi &= \frac{J_1^{(\psi)}}{r^2}\sin^2\theta\,, && (4.86) \end{aligned}$$

where $2J^2 = \left[\left(J_1^{(\phi)}\right)^2 + \left(J_1^{(\psi)}\right)^2\right]$. The asymptotic behavior of this solution is

$$\begin{aligned} f &\to \frac{\mathcal{Q}_0}{r^2} + \beta\frac{J^2}{12r^6} \quad (\text{as } r \to \infty) \\ f &\to \frac{\beta J^2}{4\mathcal{Q}_H^{(2)} r^4} \quad (\text{as } r \to 0)\,. && (4.87) \end{aligned}$$

The ADM mass is given by

$$M_{\mathrm{ADM}} = \frac{\pi}{4G_5}(\mathcal{Q}_0 + \mathcal{Q}_H^{(2)})\,. \qquad (4.88)$$

Although $r = 0$ is not a singularity, it is not a horizon because it is a timelike hypersurface. In fact, setting $J_1^{(\phi)} = -J_1^{(\psi)} = J$ (a supersymmetric spacetime), we find the surface area of r=constant as

$$\begin{aligned} A(r) &= 4\pi r^3 \int d\theta \cos\theta \sin\theta \left[H_2(1+f) - \frac{J^2}{8r^6}\right]^{1/2} \\ &\sim \pi \int d\theta \cos\theta\sin\theta\, |J| \left(\beta - \frac{1}{2}\right)^{1/2} \quad (\text{as } r \to 0)\,. && (4.89) \end{aligned}$$

This value becomes imaginary because $\beta < 1/2$. When $\beta = 1/2$, i.e., $B_i^A = 0$, the surface area (the entropy) vanishes.

This solution is also similar to the Brinkmann wave solution if $\beta > 0$. For $\beta = 0$, we have already discussed in the previous subsection. In the case of $\beta < 0$, $(1 + f)$ vanishes at finite radius $r = r_S (> 0)$, and then there exists a naked singularity.

4.4 Hyperelliptical Coordinates

First we adopt the hyperelliptical coordinates (ξ, η, ϕ, ψ), which are defined by the following transformation:

$$x_1 + ix_2 = R\cosh\xi\cos\eta e^{i\phi}, \quad x_3 + ix_4 = R\sinh\xi\sin\eta e^{i\psi}\,, \tag{4.90}$$

where R is a constant, $\xi \geq 0$, $0 \leq \eta \leq \pi$, and $0 \leq \phi, \psi \leq 2\pi$.

The line element is given by

$$ds^2_{\mathbb{E}^4} = R^2\left[(\sinh^2\xi + \sin^2\eta)(d\xi^2 + d\eta^2) + \cosh^2\xi\cos^2\eta d\phi^2 + \sinh^2\xi\sin^2\eta d\psi^2\right]\,. \tag{4.91}$$

Eq. (4.41) can be written with $A = 2, 5$ by

$$\frac{1}{\sinh\xi\cosh\xi}\partial_\xi(\sinh\xi\cosh\xi\partial_\xi H_A) + \frac{1}{\sin\eta\cos\eta}\partial_\eta(\sin\eta\cos\eta\partial_\eta H_A) = 0\,. \tag{4.92}$$

Setting $H_A = h_A(\xi)j_A(\eta)$, we find two ordinary differential equations:

$$\frac{1}{\sinh\xi\cosh\xi}\frac{d}{d\xi}\left(\sinh\xi\cosh\xi\frac{dh_A}{d\xi}\right) - Mh_A = 0, \tag{4.93}$$

$$\frac{1}{\sin\eta\cos\eta}\frac{d}{d\eta}\left(\sin\eta\cos\eta\frac{dj_A}{d\eta}\right) + Mj_A = 0\,, \tag{4.94}$$

where M is a separation constant. Using new variables $\rho = \cosh 2\xi$ and $\mu = \cos 2\eta$, these equations are rewritten by the Legendre equation as

$$\frac{d}{d\rho}\left((\rho^2 - 1)\frac{dh_A}{d\rho}\right) - \frac{M}{4}h_A = 0, \tag{4.95}$$

$$\frac{d}{d\mu}\left((1 - \mu^2)\frac{dj_A}{d\mu}\right) + \frac{M}{4}j_A = 0\,. \tag{4.96}$$

The regularity condition on the symmetric axis gives $M = 4\ell(\ell + 1) \quad (\ell = 0, 1, 2, \cdots)$. A general solution for H_A is then

$$H_A = \sum_{\ell=0}^{\infty}\left[g_\ell^{(A)}P_\ell(\cosh 2\xi) + h_\ell^{(A)}Q_\ell(\cosh 2\xi)\right]P_\ell(\cos 2\eta)\,, \tag{4.97}$$

where $Q_\ell(z)$ is the second kind Legendre function, and $g_\ell^{(A)}$ and $h_\ell^{(A)}$ are arbitrary constants.

From the condition of asymptotically flatness ($H_A \to 1$ as $r \to \infty$), H_A is given by

$$H_A = 1 + \sum_{\ell=0}^{\infty} h_\ell^{(A)}Q_\ell(\cosh 2\xi)P_\ell(\cos 2\eta) \quad (A = 2, 5)\,, \tag{4.98}$$

because $Q_\ell(z)$ vanishes at $z=\infty$. The explicit form for $\ell = 0, 1, 2$ is as follows:

$$\begin{aligned} Q_0(z) &= -\frac{1}{2}\ln\left(\frac{z+1}{z-1}\right) \\ Q_1(z) &= \frac{z}{2}\ln\left(\frac{z+1}{z-1}\right) - 1 \\ Q_2(z) &= \frac{1}{4}\left(3z^2-1\right)\ln\left(\frac{z+1}{z-1}\right) - \frac{3z}{2}\,. \end{aligned} \tag{4.99}$$

The solution of H_A with the lowest moment ($\ell = 0$) is

$$H_A = 1 + h_0^{(A)} \ln\left(\tanh\xi\right)\,. \tag{4.100}$$

If we define a charge $\mathcal{Q}_H^{(A)}$ by the asymptotic behavior of H_A as $H_A \to 1 + \mathcal{Q}_H^{(A)}/r^2$, we find that

$$h_0^{(A)} = -\frac{2\mathcal{Q}_H^{(A)}}{R^2}\,, \tag{4.101}$$

because $\ln(\tanh\xi) \sim -2/e^{2\xi} \approx -R^2/(2r^2)$.

Now we solve Eqs. (4.42) and (4.43), which are

$$\coth\xi\partial_\xi(\tanh\xi\partial_\xi\mathcal{A}_\phi) + \cot\eta\partial_\eta(\tan\eta\partial_\eta\mathcal{A}_\phi) = 0, \tag{4.102}$$

$$\tanh\xi\partial_\xi(\coth\xi\partial_\xi\mathcal{A}_\psi) + \tan\eta\partial_\eta(\cot\eta\partial_\eta\mathcal{A}_\psi) = 0\,. \tag{4.103}$$

Setting $\mathcal{A}_\phi = a_\phi(\xi)b_\phi(\eta)$ and $\mathcal{A}_\psi = a_\psi(\xi)b_\psi(\eta)$, we obtain the following ordinary differential equations:

$$\frac{d^2a_\phi}{d\rho^2} + \frac{1}{\rho-1}\frac{da_\phi}{d\rho} - \frac{K}{\rho^2-1}a_\phi = 0, \tag{4.104}$$

$$\frac{d^2b_\phi}{d\mu^2} - \frac{1}{1-\mu}\frac{db_\phi}{d\mu} + \frac{K}{1-\mu^2}b_\phi = 0, \tag{4.105}$$

$$\frac{d^2a_\psi}{d\rho^2} + \frac{1}{\rho+1}\frac{da_\psi}{d\rho} - \frac{L}{\rho^2-1}a_\psi = 0, \tag{4.106}$$

$$\frac{d^2b_\psi}{d\mu^2} + \frac{1}{1+\mu}\frac{db_\psi}{d\mu} + \frac{L}{1-\mu^2}b_\psi = 0\,, \tag{4.107}$$

where $\rho = \cosh 2\xi$ and $\mu = \cos 2\eta$, and K and L are separation constants.

Eqs. (4.105) and (4.107) are the same as Eqs (4.59) and (4.61). Then we obtain the angular solutions by hypergeometric functions as $b_\phi = F\left(-m, m, 1, (1-\mu)/2\right)$ and $b_\psi = F\left(-n, n, 1, (1+\mu)/2\right)$. We have set $K = m^2$ and $L = n^2$ ($m, n = 1, 2, \ldots$) from regularity conditions on the symmetric axis. The solutions for Eqs. (4.104) and (4.106) are also given by the hypergeometric functions. Imposing the asymptotically flatness condition at infinity ($\xi \to \infty$), we find the following solutions:

$$\mathcal{A}_\phi = \sum_{m=1}^{\infty} b_m^{(\phi)} \sinh^{-2m}\xi F\left(m, m, 1+2m, -\sinh^{-2}\xi\right) F(-m, m, 1, \sin^2\eta) \tag{4.108}$$

$$\mathcal{A}_\psi = \sum_{n=1}^{\infty} b_n^{(\psi)} \cosh^{-2n}\xi F\left(n, n, 1+2n, \cosh^{-2}\xi\right) F(-n, n, 1, \cos^2\eta)\,. \tag{4.109}$$

Here we show some hypergeometric functions, which we use later, explicitly:

$$\begin{aligned} F(1,1,3,z) &= \frac{2}{z^2}\left[z + (1-z)\ln(1-z)\right] \\ F(1,2,3,z) &= -\frac{2}{z^2}\left[z + \ln(1-z)\right]\,. \end{aligned} \tag{4.110}$$

If $\beta = 0$ (two charged branes or neutral branes with appropriately chosen current strength $\lambda_{A''}$), the solution of f is given by a harmonic function, which is

$$f = \sum_{\ell=0}^{\infty} c_\ell Q_\ell(\cosh 2\xi) P_\ell(\cos 2\eta)\,, \tag{4.111}$$

where

$$c_\ell = -\frac{2\mathcal{Q}_\ell}{R^2}\,. \tag{4.112}$$

The lowest moment solution ($\ell = 0, m = n = 1$) in this case is

$$H_A = 1 - \frac{2\mathcal{Q}_H^{(A)}}{R^2}\ln(\tanh\xi) \quad (A = 2, 5) \tag{4.113}$$

$$f = -\frac{2\mathcal{Q}_0}{R^2}\ln(\tanh\xi) \tag{4.114}$$

$$\mathcal{A}_\phi = -2J_1^{(\phi)}\left[1 + 2\cosh^2\xi\ \ln(\tanh\xi)\right]\cos^2\eta \tag{4.115}$$

$$\mathcal{A}_\psi = 2J_1^{(\psi)}\left[1 + 2\sinh^2\xi\ \ln(\tanh\xi)\right]\sin^2\eta\,, \tag{4.116}$$

where $\mathcal{Q}_H^{(2)}, \mathcal{Q}_H^{(5)}, \mathcal{Q}_0$ are charges and $J_1^{(\phi)}, J_1^{(\psi)}$ are angular momentum. We can show that this spacetime is supersymmetric if $J_1^{(\phi)} = -J_1^{(\psi)}$, which is the same condition as that for the BMPV black hole solution.

The ADM mass of this object is

$$M_{\rm ADM} = \frac{\pi}{4G_5}\left(\mathcal{Q}_0 + \mathcal{Q}_H^{(2)} + \mathcal{Q}_H^{(5)}\right)\,. \tag{4.117}$$

H_A and $(1+f)$ diverge at $\xi = 0$, which may correspond to the horizon. Calculating the Kretschmann curvature invariant, we show that it is a naked singularity. Therefore, this solution does not provide a black hole spacetime, but instead, describes the spacetime of a rotating singular disk.

In the case of $\beta \neq 0$, in order to obtain the solution for f, we have to expand f and the source term $\tilde{\mathcal{S}}$ by the Legendre function $P_\ell(\cos 2\eta)$, just as in the case of the previous hyperspherical coordinates (Eqs. (4.78) and (4.78). $\tilde{\mathcal{S}}$ is defined by

$$\begin{aligned}\tilde{\mathcal{S}}(\xi,\eta) &\equiv R^2(\sinh^2\xi+\sin^2\eta)\,\mathcal{S}(\xi,\eta) = \frac{\beta R^2(\sinh^2\xi+\sin^2\eta)}{8H_2H_5}\mathcal{F}_{ij}\mathcal{F}^{ij} \\ &= \frac{\beta}{4R^2H_2H_5}\left[\frac{(\partial_\xi\mathcal{A}_\phi)^2+(\partial_\eta\mathcal{A}_\phi)^2}{\cosh^2\xi\cos^2\eta}+\frac{(\partial_\xi\mathcal{A}_\psi)^2+(\partial_\eta\mathcal{A}_\psi)^2}{\sinh^2\xi\sin^2\eta}\right]. \quad (4.118)\end{aligned}$$

We then have

$$\frac{d}{d\rho}\left((\rho^2-1)\frac{df_\ell}{d\rho}\right)-\ell(\ell+1)f_\ell = \frac{\tilde{\mathcal{S}}_\ell}{4}\,, \quad (4.119)$$

where $\rho = \cosh 2\xi$.

Let us show one concrete example, which is the lowest moment solution ($m = n = 1$). Setting $H_2 = H_5 = 1$ (Case (2c) in Table 4.1) and

$$\mathcal{A}_\phi = -2J_1^{(\phi)}\left[1+2\cosh^2\xi\;\ln(\tanh\xi)\right]\cos^2\eta \quad (4.120)$$

$$\mathcal{A}_\psi = 2J_1^{(\psi)}\left[1+2\sinh^2\xi\;\ln(\tanh\xi)\right]\sin^2\eta\,, \quad (4.121)$$

we find

$$\tilde{\mathcal{S}}_0(\rho) = \frac{8\beta J^2}{R^2}\left[\frac{2\rho}{\rho^2-1}+2\ln\left(\frac{\rho-1}{\rho+1}\right)+\frac{\rho}{2}\left(\ln\left(\frac{\rho-1}{\rho+1}\right)\right)^2\right] \quad (4.122)$$

$$\tilde{\mathcal{S}}_1(\rho) = \frac{8\beta J^2}{R^2}\left[\frac{2}{\rho^2-1}-\frac{1}{2}\left(\ln\left(\frac{\rho-1}{\rho+1}\right)\right)^2\right], \quad (4.123)$$

where $2J^2 = (J_1^{(\phi)})^2+(J_1^{(\psi)})^2$. Integrating Eq. (4.119), we find the exact solution as

$$f(\xi,\eta) = f_0(\xi)+f_1(\xi)P_1(\cos 2\eta)\,, \quad (4.124)$$

with

$$f_0(\xi) = -\frac{2\mathcal{Q}_0}{R^2}\ln(\tanh\xi)+\frac{2\beta J^2}{R^2}\cosh 2\xi\,[\ln(\tanh\xi)]^2 \quad (4.125)$$

$$f_1(\xi) = \frac{2\mathcal{Q}_1}{R^2}\left[1+\cosh 2\xi\ln(\tanh\xi)\right]-\frac{\beta J^2}{2R^2}\left[\ln(\tanh\xi)\right]^2. \quad (4.126)$$

The ADM mass is

$$M_{\rm ADM} = \frac{\pi}{4G_5}\mathcal{Q}_0 \,. \qquad (4.127)$$

Although this is an exact solution, it is very complicated. Unless $\beta = 0$, the horizon, even if it exists, is not described by a surface of ξ= constant.

Note that although this solution is very complicated, it is still supersymmetric if $J_1^{(\phi)} = -J_1^{(\psi)}$.

4.5 Hyperpolorical Coordinates

Our next example is the hyperpolorical coordinates (ξ, η, ϕ, ψ), which are defined by the transformation

$$x_1 + ix_2 = \frac{R\sinh\xi}{\cosh\xi - \cos\eta}e^{i\psi}, \quad x_3 + ix_4 = \frac{R\sin\eta}{\cosh\xi - \cos\eta}e^{i\phi} \,, \qquad (4.128)$$

where $\xi \geq 0$, $0 \leq \eta \leq \pi$, and $0 \leq \phi, \psi \leq 2\pi$. This coordinates could be used to describe a ring topology, which is also same symmetry as [92, 93]. In this case, the infinity corresponds to $\xi = 0$, which also describes one of the symmetric axis.

The line element is given by

$$ds^2_{\mathbb{E}^4} = \frac{R^2}{(\cosh\xi - \cos\eta)^2}(d\xi^2 + \sinh^2\xi d\psi^2 + d\eta^2 + \sin^2\eta d\phi^2) \,. \qquad (4.129)$$

With this coordinate system, Eq. (4.41) is written as

$$\frac{1}{\sinh\xi}\partial_\xi\left(\frac{\sinh\xi}{(\cosh\xi - \cos\eta)^2}\partial_\xi H_A\right) + \frac{1}{\sin\eta}\partial_\eta\left(\frac{\sin\eta}{(\cosh\xi - \cos\eta)^2}\partial_\eta H_A\right) = 0 \,. \quad (4.130)$$

Using new variable $\tilde{H}_A$, which is defined by $H_A(\xi, \eta) = 1 + (\cosh\xi - \cos\eta)\tilde{H}_A(\xi, \eta)$, Eq. (4.130) is rewritten as

$$\partial^2_\xi\tilde{H}_A + \coth\xi\partial_\xi\tilde{H}_A + \partial^2_\eta\tilde{H}_A + \cot\eta\partial_\eta\tilde{H}_A = 0 \,. \qquad (4.131)$$

Setting $\tilde{H}_A = \tilde{h}_A(\xi)\tilde{j}_A(\eta)$, we can separate the variables and find the following two ordinary differential equations:

$$(\rho^2 - 1)\frac{d^2\tilde{h}_A}{d\rho^2} + 2\rho\frac{d\tilde{h}_A}{d\rho} - M\tilde{h}_A = 0, \qquad (4.132)$$

$$(1 - \mu^2)\frac{d^2\tilde{j}_A}{d\mu^2} - 2\mu\frac{d\tilde{j}_A}{d\mu} + M\tilde{j}_A = 0 \,, \qquad (4.133)$$

where $\rho = \cosh\xi$ and $\mu = \cos\eta$, and M is a separation constant.

We find that the general solution is described by the Legendre functions as

$$H_A = 1 + (\cosh\xi - \cos\eta)\sum_{\ell=0}^{\infty}\left[h_\ell^{(A)}P_\ell(\cosh\xi) + g_\ell^{(A)}Q_\ell(\cosh\xi)\right]P_\ell(\cos\eta)\,, \quad (4.134)$$

where the separation constant M is given by an integer ℓ as $M = \ell(\ell+1)$ because of the regularity on the symmetric axis. $g_\ell^{(A)}$ and $h_\ell^{(A)}$ are arbitrary constants.

The asymptotically flatness condition yields

$$H_A = 1 + (\cosh\xi - \cos\eta)\sum_{\ell=0}^{\infty}h_\ell^{(A)}P_\ell(\cosh\xi)P_\ell(\cos\eta)\,. \quad (4.135)$$

Since $r^2 = R^2(\cosh\xi + \cos\eta)/(\cosh\xi - \cos\eta)$, looking at the asymptotic behavior at infinity, we find $(\cosh\xi - \cos\eta) \sim 2R^2/r^2$ as $r \to \infty(\xi, \eta \to 0)$. This gives a relation between the coefficient $h_0^{(A)}$ and charge $\mathcal{Q}_H$ as

$$h_0^{(A)} = \frac{\mathcal{Q}_H^{(A)}}{2R^2}\,. \quad (4.136)$$

Now we discuss Eqs. (4.42) and (4.43), which are

$$\frac{1}{\sinh\xi}\partial_\xi(\sinh\xi\partial_\xi\mathcal{A}_\phi) + \sin\eta\partial_\eta\left(\frac{1}{\sin\eta}\partial_\eta\mathcal{A}_\phi\right) = 0 \quad (4.137)$$

$$\sinh\xi\partial_\xi\left(\frac{1}{\sinh\xi}\partial_\xi\mathcal{A}_\psi\right) + \frac{1}{\sin\eta}\partial_\eta(\sin\eta\partial_\eta\mathcal{A}_\psi) = 0\,. \quad (4.138)$$

Setting $\mathcal{A}_\phi = a_\phi(\xi)b_\phi(\eta)$ and $\mathcal{A}_\psi = a_\psi(\xi)b_\psi(\eta)$, we obtain the following ordinary differential equations:

$$\frac{d^2a_\phi}{d\rho^2} + \frac{2\rho}{\rho^2-1}\frac{da_\phi}{d\rho} - \frac{K}{\rho^2-1}a_\phi = 0, \quad (4.139)$$

$$\frac{d^2b_\phi}{d\mu^2} + \frac{K}{1-\mu^2}b_\phi = 0, \quad (4.140)$$

$$\frac{d^2a_\psi}{d\rho^2} - \frac{L}{\rho^2-1}a_\psi = 0, \quad (4.141)$$

$$\frac{d^2b_\psi}{d\mu^2} - \frac{2\mu}{1-\mu^2}\frac{db_\psi}{d\mu} + \frac{L}{1-\mu^2}b_\psi = 0\,, \quad (4.142)$$

where $\rho = \cosh\xi$ and $\mu = \cos\eta$, and K and L are separation constants.

The solutions for Eqs. (4.140) and (4.142) are given by the Legendre functions. We set $K = m(m+1)$ and $L = n(n+1)$ $(m, n = 1, 2, \cdots)$ because of regularity conditions on the symmetric axis.

The asymptotically flatness condition yields

$$\mathcal{A}_\phi = \sum_{m=1}^{\infty} \frac{b_m^{(\phi)}}{m+1} P_m(\cosh\xi)\left[\cos\eta\, P_m(\cos\eta) - P_{m-1}(\cos\eta)\right] \tag{4.143}$$

$$\mathcal{A}_\psi = \sum_{n=1}^{\infty} \frac{b_n^{(\psi)}}{n+1} \left[\cosh\xi\, P_n(\cosh\xi) - P_{n-1}(\cosh\xi)\right] P_n(\cos\eta)\,, \tag{4.144}$$

where $b_m^{(\phi)}$ and $b_n^{(\psi)}$ are arbitrary constants.

When $\beta = 0$, f is given by the Legendre functions just as H_A, i.e.,

$$f = (\cosh\xi - \cos\eta) \sum_{\ell=0}^{\infty} c_\ell P_\ell(\cosh\xi) P_\ell(\cos\eta)\,, \tag{4.145}$$

where c_ℓ's are arbitrary constants. For the lowest moment solution, we find

$$H_A(\xi,\eta) = 1 + \frac{\mathcal{Q}_H^{(A)}}{2R^2}(\cosh\xi - \cos\eta), \tag{4.146}$$

$$f(\xi,\eta) = \frac{\mathcal{Q}_0}{2R^2}(\cosh\xi - \cos\eta), \tag{4.147}$$

$$\mathcal{A}_\phi = J_1^{(\phi)} \cosh\xi \sin^2\eta \tag{4.148}$$

$$\mathcal{A}_\psi = J_1^{(\psi)} \sinh^2\xi \cos\eta\,. \tag{4.149}$$

The self-dual condition for supersymmetry implies $J_1^{(\phi)} = -J_1^{(\psi)}$.

In this spacetime, there is no horizon, but rather, a singularity at $\xi = \infty$, which locates at a ring with a radius R in the flat 4D Euclidian space. Then this describes the geometry of a ring singularity.

To solve the equation for f in the case of $\beta \neq 0$, we again expand f and the source term $\tilde{\mathcal{S}}$ by the Legendre functions as

$$f(\xi,\eta) \equiv (\rho - \mu) \sum_{\ell=0}^{\infty} \tilde{f}_\ell(\rho) P_\ell(\mu) \tag{4.150}$$

$$\begin{aligned}
\tilde{\mathcal{S}}(\xi,\eta) &\equiv \frac{R^2}{(\cosh\xi - \cos\eta)^3}\,\mathcal{S}(\xi,\eta) = \frac{\beta R^2}{8H_2H_5(\cosh\xi - \cos\eta)^3}\mathcal{F}_{ij}\mathcal{F}^{ij} \\
&= \frac{\beta(\cosh\xi - \cos\eta)}{4R^2 H_2 H_5}\left[\frac{1}{\sin^2\eta}\left((\partial_\xi\mathcal{A}_\phi)^2 + (\partial_\eta\mathcal{A}_\phi)^2\right) + \frac{1}{\sinh^2\xi}\left((\partial_\xi\mathcal{A}_\psi)^2 + (\partial_\eta\mathcal{A}_\psi)^2\right)\right] \\
&= \sum_{\ell=0}^{\infty} \tilde{\mathcal{S}}_\ell(\rho) P_\ell(\mu)\,,
\end{aligned} \tag{4.151}$$

where $\rho = \cosh\xi$ and $\mu = \cos\eta$.

We obtain the equation for $f_\ell(\rho)$ for each ℓ as

$$(\rho^2-1)\frac{d^2\tilde{f}_\ell}{d\rho^2}+2\rho\frac{d\tilde{f}_\ell}{d\rho}-\ell(\ell+1)\tilde{f}_\ell=\tilde{\mathcal{S}}_\ell(\rho)\,, \tag{4.152}$$

where $\rho=\cosh\xi$.

Setting $H_2=H_5=1$ (Case (2c) in Table 4.1) and

$$\mathcal{A}_\phi = J_1^{(\phi)}\cosh\xi\sin^2\eta \tag{4.153}$$
$$\mathcal{A}_\psi = J_1^{(\psi)}\sinh^2\xi\cos\eta\,, \tag{4.154}$$

we show the lowest moment solution here. We have now

$$\begin{aligned}
\tilde{\mathcal{S}}_0 &= \frac{\beta J^2}{3R^2}\rho\left(3\rho^2-1\right)\\
\tilde{\mathcal{S}}_1 &= -\frac{\beta J^2}{5R^2}\left(7\rho^2-1\right)\\
\tilde{\mathcal{S}}_2 &= \frac{\beta J^2}{3R^2}\rho\left(3\rho^2+1\right)\\
\tilde{\mathcal{S}}_3 &= -\frac{\beta J^2}{5R^2}\left(3\rho^2+1\right)\,,
\end{aligned} \tag{4.155}$$

where $2J^2\equiv[(J_1^{(\phi)})^2+(J_1^{(\psi)})^2]$, and then find general solutions as

$$\begin{aligned}
\tilde{f}_0(\rho) &= c_0P_0(\rho)+d_0Q_0(\rho)+\frac{\beta J^2}{24R^2}\left[2\rho(\rho^2+1)+\ln\left(\frac{\rho-1}{\rho+1}\right)\right]\\
\tilde{f}_1(\rho) &= c_1P_1(\rho)+d_1Q_1(\rho)-\frac{\beta J^2}{40R^2}\rho\left[14\rho+5\ln\left(\frac{\rho-1}{\rho+1}\right)\right]\\
\tilde{f}_2(\rho) &= c_2P_2(\rho)+d_2Q_2(\rho)-\frac{\beta J^2}{24R^2}\rho\left[-2\rho(2\rho^2-1)-(3\rho^2-1)\ln\left(\frac{\rho-1}{\rho+1}\right)\right]\\
\tilde{f}_3(\rho) &= c_3P_3(\rho)+d_3Q_3(\rho)+\frac{\beta J^2}{10R^2}\rho^2\,.
\end{aligned} \tag{4.156}$$

Imposing the regularity ($\tilde{f}_0$: finite, $\tilde{f}_\ell=0$ for $\ell\geq 1$) at infinity and on the axis ($\rho=1$), we can fix the coefficients c_ℓ and d_ℓ ($\ell=0,1,2,3$) except for c_0. We obtain an exact solution as

$$f(\xi,\eta)=(\cosh\xi-\cos\eta)\sum_{\ell=0}^{3}\tilde{f}_\ell(\cosh\xi)P_\ell(\cos\eta)\,, \tag{4.157}$$

with

$$
\begin{aligned}
\tilde{f}_0(\rho) &= \frac{\beta J^2}{12R^2}(\rho-1)(\rho^2+\rho+2)+\frac{\mathcal{Q}_0}{2R^2} \\
\tilde{f}_1(\rho) &= -\frac{\beta J^2}{20R^2}(\rho-1)(7\rho+5) \\
\tilde{f}_2(\rho) &= +\frac{\beta J^2}{12R^2}(\rho-1)(2\rho^2+5\rho+1) \\
\tilde{f}_3(\rho) &= -\frac{\beta J^2}{20R^2}\rho(\rho-1)(5\rho+3)\,,
\end{aligned}
\tag{4.158}
$$

where $\mathcal{Q}_0$ is an arbitrary charge. The ADM mass is given by

$$
M_{\mathrm{ADM}} = \frac{\pi}{4G_5}\mathcal{Q}_0\,. \tag{4.159}
$$

This exact solution is also very complicated, but supersymmetric if $J_1^{(\phi)} = -J_1^{(\psi)}$. The horizon, even if it exists, is not described by a surface of ξ=constant.

Chapter 5

Cosmological Solution

In this chapter we consider the cosmological solutions form the $N2 \perp N5$ null dependent brane solutions.

To avoid the No Go Theorem of inflationary universe in Supergravity with diagonal Compactification [95], we have assumed the null like symmetry. We will show that it is one of the possible way to get the inflation universe.

We also can not apply the Kalza-Klein Compactification directly from eleven dimension to four dimension, because the metric components are depend on y_1 direction which we would like to compactify finally. Thus we first compactify the five internal space $\{y_2, \cdots, y_6\}$, and then we will show that if we consider the inflationary solutions in four dimension, we will find a dynamically Compactification of y_1 direction.

5.1 Inflation Universe from Intersecting N-branes

We first compactify the $p-1$ dimension denoted by y_α $(\alpha = 2, \cdots, p)$ using the Kaluza-Klein Compactification. Since in the (d+1)-dimensional Einstein frame, warp factor Ω can be determined as

$$\Omega = \prod_A H_A^{-\frac{D-q_A-3-(q_A+1)(d-2)}{(d-1)\Delta_A}} . \tag{5.1}$$

Therefore the (d+1)-dimensional metric can be written as

$$ds^2 = \Xi^{d-2} 2du(dv + f du) + \Xi^{-1} \sum_{i=1}^{d-1} dx_i^2 , \tag{5.2}$$

where Ξ is defined as

$$\Xi = \prod_A H_A^{-\frac{2(D-2)}{(d-1)\Delta_A}} . \tag{5.3}$$

To remember the components of the metric is satisfy that

$$H_A = \alpha_A + g_A(u)\,, \quad f = f(u)\,, \tag{5.4}$$

where $g_A(u)$ and $f(u)$ are arbitrary functions of u.

We would like to get the four-dimenisonal inflation universe, so we choose the $N2 \perp N5$ brane solution (3.121), and the metric can be written as

$$\begin{aligned} ds^2 &= (H_2H_5)^{-2/3}2du(dv + fdu) + (H_2H_5)^{1/3}\sum_{i=1}^{3} dx_i^2 \\ &= -(H_2H_5)^{-2/3}(1+f)^{-1}dt^2 + (H_2H_5)^{1/3}\sum_{i=1}^{3} dx_i^2 \\ &\quad + (H_2H_5)^{-2/3}(1+f)\left(dy_1 - \frac{f}{1+f}dt\right)^2 . \end{aligned} \tag{5.5}$$

We now change the coordinate t for the cosmic time τ, and the y_1 for the bulk space y as

$$d\tau = (H_2H_5(1+f))^{-1/4}dt\,, \quad dy = dy_1 - \frac{f}{1+f}dt\,, \tag{5.6}$$

then we find the metric becomes

$$ds^2 = \Omega^2\left(-d\tau^2 + a^2\sum_{i=1}^{3} dx_i^2 + b^2dy^2\right)\,, \tag{5.7}$$

where a four dimensional scale factor a, a warp factor b for the bulk space radius y and a conformal factor Ω are defined by

$$a \equiv \frac{dt}{d\tau} = (H_2H_5(1+f))^{1/4} \tag{5.8}$$

$$b = (H_2H_5)^{-1/4}(1+f)^{3/4} \tag{5.9}$$

$$\Omega = (H_2H_5)^{-1/12}(1+f)^{-1/4}\,. \tag{5.10}$$

In order to solve the equations (5.4), we must choose a boundary condition or initial data condition. However the initial data of the universe is not well known, because of the energy scale is too high. In this regime, Supergravity is also not allow to apply it, thus we choose different condition for the solutions. This condition is that bulk space y must be compactify at the end of inflation, which can be written by

$$b/a = (H_2H_5)^{-1/2}(1+f)^{1/2} \to 0\,. \tag{5.11}$$

We assume that $g_A(u)$ can be written by polynomial function, and we are only interested in the dominant term of contribution from H_A and f, thus we can write down $H_A \sim u^s$, where we also assume all H_A functions give the same contributions. In this case if the Supersymmetry remain $1/8$, the function f with constant ϵ_0 can be written by

$$f = \frac{2H_2H_5}{\alpha_5 H_2 + 2\alpha_2 H_5}\left(\frac{\partial_u H_2}{H_2} + \frac{\partial_u H_5}{2H_5}\right) \sim u^{s-1}\,. \tag{5.12}$$

For the compactification condition we must choose $s > 1/5$. We also find that the scale factor can be written by $a \sim u^{(3s-1)/4}$, thus if we would like to get the expanding universe we must choose $s > 1/3$.

One interesting question about the early stage of the universe is whether the inflationary solutions are consistent with Supersymmetry. We choose $s = 5/3$, which is satisfy the all conditions we mention above, then we have a exponential expanding universe $a(\tau) \propto e^{\tau}$ is realized. We assume the breaking of Supersymmetry and the end of inflation occurs simultaneously, and the reheating temperature is obtained by a latent heat from breaking Supersymmetry. After the Supersymmetry breaking, the BPS condition for f is despairs, and we also suppose Moduli stabilization $H_2H_5 = 1 + f$, which means that b is fixed. The extra-dimensianl bulk space can be integrated out, we finally find the Robertson-Walker metric as

$$ds^2 = -d\tau^2 + a^2\sum_{i=1}^{3} dx_i^2\,, \tag{5.13}$$

where a is given by $a = (H_2H_5)^{1/2}$. Unfortunately in this case the scale factor becomes $a(\tau) \propto \tau^{-5/3}$, and this is unphysical. It might be occur we does not care about the mechanism of Supersymmetry breaking, because if we consider $1 + f \propto u^{-2/3}$, the bulk space $b \propto u^{-3}$ and the scale factor $a(\tau) \propto \tau^{1/2}$, thus we find naturally connection from inflation universe to radiational universe via reheating. In this case we can not stabilized the moduli naively, however the contribution from the moduli b is too small at the radiation dominant era. Therefore if we does not worry about it, we might get a realistic universe model from N-branes.

It is worthwhile to relate the above cosmological solution with other solutions based on other types of string theory. When it comes to $y_2, \ldots, y_7$ coordinates, which are compactified in the above analysis, following to the similar idea, it can be shown that the volume element of y_2 coordinate shrinks more rapidly than that of $y_3, \ldots, y_7$ coordinates. If we comapctify only y_2 coordinate in 11-dimensional M-theory, we can find the $D2$ and $NS5$-brane's bound state in type IIA string theory. Furthermore, using the T-duality transformation on y^3 direction, the $D3$- and $NS5$-brane's bound state is obtained. It is shown that the $D3$-brane is rolling in"throat geometry" on the $NS5$ background [97], and the corresponding cosmological solutions are already provided by [98], which is related to rolling tachyon given by [99], from the point of view of the string theory.

Chapter 6

Concluding Remarks

In this thesis, we have studied about stationary and null dependent solutions in eleven- and ten-dimensional Supergravity, which are the low energy effective theories of Super String and M-theory. Assuming a BPS type relation, which is almost the same as a integrable condition, between the first-order derivatives of metric functions, we have shown how to construct a stationary black brane solution with a traveling wave, and a null dependent N-brane solution with a traveling wave.

In black brane case, we consider two types of intersecting branes: (1) charged branes and (2) neutral branes with a current. The solutions are given by harmonic functions H_A and $\mathcal{A}_i$ plus a wave metric f which satisfies the Poisson equation for $\beta \neq 0$ (Cases (2a)-(2c) in Table II) or the Laplace equation for $\beta = 0$ (Case (1) and Cases (2a)-(2c) with specific values for λ_A in Table 4.2). Since those differential equations are linear and independent except for the Poisson equation for f, we can easily construct general solutions by superposition of harmonic functions.

In N-brane case, we find a only one possible way to construct the four-dimensional universe model from $N2 \perp N5$ intersecting branes. The gauge field and all metric components are given by almost harmonic function H_A and arbitrary function of $f(u)$. However if f satisfy the Supersymmetric condtion, f is also written by H_A.

After the Kaluza-Klein compactification, we find a five-dimenisonal black hole solutions from the black brane solution, and a four-dimensional universe with extra one-dimensional bulk space from the N-brane solution. The black hole solutions are written by flat base space, thus we choose some symmetry on the base space, and solve the equations to find the some kinds of black holes. Cosmological solutions with bulk space, the metric function is almost arbitrary, thus we choose it by a requirement of our purpose.

In black hole case, using the hyperspherical coordinate system for our conformally flat base space, we show that these solutions include the BMPV black hole and the Brinkmann wave solution, and those extension to non-BPS ones. We have also found new solutions which are similar to the Brinkmann wave.

We have proved that the solutions preserve the 1/8 supersymmetry if $\mathcal{F}_{ij}$ is self-dual. All solutions found in the hyperspherical coordinates preserve the 1/8 supersymmetry if the angular momenta satisfy some relation (e.g., $J_\phi = -J_\psi$).

We also discuss non-spherical black brane solutions (e.g., a ring topology and an elliptical shape solution) by use of hyperelliptical and hyperpolorical coordinates. Unfortunately, we could not find any regular solutions, but in stead, solutions with a naked singularity. In particular, even if we use hyperpolorical coordinates, we do not obtain a black ring solution.

However we know that it is constructed in M theory or Type II theory by different approaches [100, 101]. To find such a black ring solution in our approach, we may have to generalize some parts of anstaz. When we rewrite a supersymmetric black ring solution in our null coordinate system [102], we find another gravi-electromagnetic field in addition to $\mathcal{A}_j$, which suggests that we should replace $\theta^{\hat{u}} = e^{\xi}du$ with $\theta^{\hat{u}} = e^{\xi}(du + \mathcal{B}_j dx^j)$. As for the configuration of branes, we have two known cases[100, 101]: One is three charged M2-branes and three m5-diploles and the other is D1-D5 branes with a pp-wave and KK dipoles. The only possible way to get the black ring solutions is to consider the M2$\perp$M2$\perp$M2 intersecting brane in time-like Killing case. In this sense, the Chern-Simons term doesn't vanishing automatically, thus we find Poisson equations for the harmonic function H_A. This will give the same solutions for the Supersymmteric Black Ring solutions we would like to get. Such an extension might provide us non-BPS type black ring solution as well.

The charges of branes of the BMPV black hole correspond to the numbers of D-brane tension. While SO(4) rotational symmetries, which describe angular momenta of the black hole, corresponds to endmorphisms in the graded algebra that rotate the fermionic generators G^i_m [47]. By this correspondence (AdS/CFT correspondence), we can discuss the properties of our solutions in the SCFT side.

Although we assume the BPS type relations for the metric, we have to solve the elliptic type differential equations if we want to find most general solutions, especially non-BPS spacetimes. For this purpose, we need a completely different approach such as a soliton technique to generate new solutions [103, 104, 105, 106].

We have found that the BPS and non-BPS rotating asymptotically flat black brane solutions, from which we may demonstrate the connections between microscopic and macroscopic states of black hole in lower dimension. In our framework, we consider a Kaluza-Klein compactification on torus form String Theory, but one may embed the BMPV type geometry form compactification from M-theory on generic Calabi-Yau spaces,which would be more interesting.

On the other hand, in cosmological case, we find inflationary solutions in four-dimension. we applied the BPS solutions to the cosmological setting by compactifying the extra coordinates, even though we have not mentioned the details of the mechanism of the compactification. The evolution of the scale factor depends on the form of H_2, H_5, and f, which are arbitrary even after imposing the BPS conditions. By choosing the functions appropriately, we obtained the exponentially expanding Universe, as well as the power-law inflationary solutions. Since these inflationary solutions are consistent with supersymmetry, it seems interesting, even though the mechanism to fix the functions is unclear at this time. Furthermore, if we concentrate on the above inflationary solutions, even without compactifying the y_1 direction at first, the compactification of the corresponding coordinate happens dynamically.

From the view point of the realistic cosmology, we cannot resist asking whether the standard big-bang Universe is recovered. One interesting scenario is the above inflationary solution becomes unstable as a result of supersymmetry breaking, and the inflation terminates. After the supersymmetry braking baryongenesis can be occurred and the dark energy and dark matter may be explained by some model of symmetry breaking [107], however we doesn't mention about it, because our main purpose is to make a general cosmological solutions in string theory.

After supersymmetry breaking, since we need not take account of the BPS conditions, all we have to consider are Eqs. (3.110), (3.111), and (3.112), which are by far milder than the BPS conditions. For example, if we choose $1 + f \propto u^{-2/3}$, the cosmic expansion law of the radiation dominated Universe is obtained. However in this case the moduli b of compact space is not stabilized, but decaying τ^{-3} order. It is also known that supersymmetry breaking generates potential heat, which describes the reheating process, even though we do not mention the details, and we would like to leave them for future work.

The fundamental problem about these solutions is the question of their stability. Still they have preserve supersymmetry, the solutions dive into the future singularity in four dimension. However using the AdS/CFT correpondence the Big crunch singularity are not singularity in CFT point of view by [108]. The supersymmetrical inflationary solutions in de Sitter solutions are stable too [38]. Therefore the presence of a singularity is not so bad only for inflationary solutions, but it is still open problems. Our solutions in the view point of string theory doesn't understand despite they have preserve supersymmetry at all, thus we must make a dynamical state in type II string theory, and compare to the solutions to the state of the string.

Acknowledgements

I would like to thank Kei-ichi Maeda for his supervision and expert guidance throughout the course of my studies. Thanks also to Shuntaro Mizuno and Nobuyoshi Ohta for their fruitful collaborations and willing helpfulness. I am also grateful to my examiners Ichiro Ohba, Tsuneaki Daishido and Shoichi Yamada for their helpful comment. My endless thanks are extended to my family for their continual love, support and encouragement. I would like to express my gratitude to the 21st century COE Program (Holistic Research and Education Centre for Physics self-organization systems) at Waseda University. Also thanks to my office-mate past and present, whose combined help, destraction and friendship have made the office such an enjoyable place to be.

Finally I deeply grateful to Yoshika Sano, who fills me with the priceless time in love and joy of my life. So I dedicate this thesis to my wife Yoshika for everlasting love, and I hope that God bless everyone who help me in the future.

Bibliography

[1] E. Witten, Nucl. Phys. B 443 (1995) 85, [hep-th/9503124].

[2] A. Strominger and C. Vafa, Phys. Lett. B 379 (1996) 99, [hep-th/9601029].

[3] G.W. Gibbons, Nucl. Phys. B 207 (1982) 337.

[4] R.C. Myers, Nucl. Phys. B 289 (1987) 701.

[5] G. W. Gibbons, and K. Maeda, Nucl. Phys. B 298 (1988) 741.

[6] C.G. Callan, R.C. Myers, and M.J. Perry, Nucl. Phys. B 311 (1988) 673.

[7] D. Garfinkle, G.T. Horowitz and A. Strominger,
Phys. Rev. D 43 (1991) 3140, Erratum-ibid. D 45 (1992) 3888.

[8] G.T. Horowitz and A. Strominger, Nucl. Phys. B 360 (1991)197.

[9] F.R. Tangherlini, Nuovo Cim. 27 (1963) 636.

[10] R.C. Myers and M.J. Perry, Ann. Phys. 172 (1986) 304.

[11] G. W. Gibbons, D. Ida and T. Shiromizu,
Phys. Rev. Lett. 89 (2002) 041101, [hep-th/0206136].

[12] H. S. Reall, Phys. Rev. D 68 (2003) 024024, [hep-th/0211290].

[13] H. Elvang, R. Emparan JHEP 0311 (2003) 035, [hep-th/0310008].

[14] R.R. Khuri, and R.C. Myers,
Fields Inst. Comm. 15 (1997) 273, [hep-th/9512137].

[15] M.J. Duff and J.X. Lu, Nucl. Phys. B 416 (1994) 301, [hep-th/9306052].

[16] M.J. Duff, J.X. Lu and C.N. Pope,
Phys. Lett. B 382 (1996) 73, [hep-th/9604052].

[17] C. G. Callan, J. M. Maldacena,
Nucl. Phys. B472 (1996) 591, [hep-th/9602043].

[18] R. Emparan and H.R. Reall,
Phys. Rev. Lett. 88 (2000) 101101, [hep-th/0110260].

[19] S. W. Hawking, Comm. Math. Phys. 43 (1975) 199.

[20] S. Gao and R. M. Wald, Phys. Rev. D 64 (2001) 084020, [gr-qc/0106071].

[21] A. Strominger, JHEP 9802 (1998) 009, [hep-th/9712251].

[22] M. Cvetic, and D. Youm, Nucl. Phys. B 453 (1995) 259, [hep-th/9505045].

[23] M. Cvetic and D. Youm, Phys. Rev. Lett. 75 (1995) 4165, [hep-th/9503082].

[24] F. Larsen and F. Wilczek, Nucl. Phys. B 458 (1996) 249, [hep-th/9506066].

[25] J. M. Maldacena, Nucl. Phys. B 477 (1996) 168, [hep-th/9605016].

[26] C. V. Johnson, R. R. Khuri and R. C. Myers,
Phys. Lett. B 378 (1996) 78, [hep-th/9603061].

[27] J. Maldacena, A. Strominger, E. Witten,
JHEP 9712 (1997) 002, [hep-th/9711053].

[28] A. A. Tseytlin and C. Vafa, Nucl. Phys. B 372 (1992) 433, [hep-th/9109048].

[29] J. Khoury, P. J. Steinhardt, D. Waldram,
Phys. Rev. D 63 (2001) 103505, [hep-th/0006069].

[30] J. Khoury, B. A. Ovrut, P. J. Steinhardt, N. Turok,
Phys. Rev. D 64 (2001) 123522, [hep-th/0103239].

[31] P. Hořava and E. Witten, Nucl. Phys. B 460 (1996) 506, [hep-th/9510209].

[32] P. Hořava and E. Witten, Nucl. Phys. B 475 (1996) 94, [hep-th/9603142].

[33] P. J. Steinherdt and N. Turok,
Phys. Rev. D 65, 126003 (2002) 443, [hep-th/0111030].

[34] A. Linde, Int. J. Mod. Phys. A 17S1 (2002) 89, [hep-th/0107176].

[35] R. Kallosh, A. Linde, S. Prokushkin, M. Shmakova,
Phys. Rev. D 65 (2002) 105016, [hep-th/0110089].

[36] T. Banks, M.Berkooz, P.J. Steinhardt,
Phys. Rev. D 52 (1995) 705, [hep-th/9501053].

[37] A. Linde, Phys. Rev. D 49 (1994) 748, [astro-ph/9307002].

[38] S. Kachru, R. Kallosh, A. Linde and S. P. Trivedi, Phys. Rev. D 68 (2003) 046005, [hep-th/0301240].

[39] P. Ramond, Phys. Rev. D 3 (1971) 2415.

[40] A. Neveu and J. H. Schwarz, Nucl. Phys. B 31 (1871) 86.

[41] M. B. Green and J. H. Schwarz, Phys. Lett.136 B (1984) 367.

[42] M. Henneaux and L. Mezincescu Phys. Lett. B 152 (1985) 340.

[43] J. M. Maldacena, Adv. Theor. Math. Phys. 2 (1998) 231; Int. J. Theor. Phys. 38 (1999) 1113, [hep-th/9711200].

[44] L. Brink and P. S. Howe, Phys. Lett. B 91 (1980) 384.

[45] B. E. W. Nilsson, Nucl. Phys. B 188 (1981) 176.

[46] E. Witten, Nucl. Phys. B 266 (1986) 245.

[47] C.A.R. Herdeiro, Nucl. Phys. B 582 (2000) 363, [hep-th/0003063].

[48] R. Argurio, F. Englert, and L. Houart, Phys. Lett. B 398 (1997) 61, [hep-th/9701042].

[49] N. Ohta, Phys. Lett. B 403 (1997) 218, [hep-th/9702164].

[50] N. Ohta, K.L. Panigrahi and Sanjay, Nucl. Phys. B 674 (2003) 306, [hep-th/0306186].

[51] Y.G. Miao and N. Ohta, Phys. Lett. B 594 (2004) 218, [hep-th/0404082].

[52] V. Frolov, W. Israel, A. Zelnikov, Phys. Rev. D72 (2005) 084031, [hep-th/0506001].

[53] E. Cremmer, B. Julia and J. Sherk, Phys. Lett. B 76 (1978) 409.

[54] M. Bertolini, P. Frè and M. Trigiante, Class. Quant. Grav. 16 (1999) 1519, [hep-th/9811251].

[55] M. Gutperle and A. Strominger, JHEP 0204 (2002) 018, [hep-th/0202210].

[56] I. I. Kogan and N. B. B. Reis, Int. J. Mod. Phys. A 16 (2001) 4567, [hep-th/0107163].

[57] A. Strominger, JHEP 0110 (2001) 034, [hep-th/0106113].

[58] C. M. Chen, D. V. Gal'tsov and M. Gutperle, Phys. Rev. D 66 (2002) 024043, [hep-th/0204071].

[59] M. Kruczenski, R. C. Myers and A. W. Peet, JHEP 0205 (2002) 039, [hep-th/0204144].

[60] S. Roy, JHEP 0208 (2002) 025, [hep-th/0205198].

[61] S. Bhattacharya and S. Roy, JHEP 0312 (2003) 015, [hep-th/0309202].

[62] K. Hashimoto, P. M. Ho and J. E. Wang, Phys. Rev. Lett. 90 (2003) 141601, [hep-th/0211090].

[63] G. Jones, A. Maloney and A. Strominger, Phys. Rev. D 69 (2004) 126008, [hep-th/0403050].

[64] G. Tasinato, I. Zavala, C. P. Burgess and F. Quevedo, JHEP 0404 (2004) 038, [hep-th/0403156].

[65] N. Ohta, Phys. Rev. Lett. 91 (2003) 061303, [hep-th/0303238].

[66] P. K. Townsend and M. N. R. Wohlfarth, Phys. Rev. Lett. 91 (2003) 061302, [hep-th/0303097].

[67] N. Ohta, Prog. Theor. Phys. 110 (2003) 269, [hep-th/0304172].

[68] K. Maeda and N. Ohta, Phys. Lett. B 597 (2004) 400, [hep-th/0405205].

[69] K. Maeda and N.Ohta, Phys. Rev. D 71 (2005) 063520, [hep-th/0411093].

[70] K. Behrndt and S. Forste, Nucl. Phys. B 430 (1994) 441, [hep-th/9403179].

[71] H. Lu, S. Mukherji, C. N. Pope and K. W. Xu, Phys. Rev. D 55 (1997) 7926, [hep-th/9610107].

[72] A. Lukas, B. A. Ovrut and D. Waldram, Nucl. Phys. B 495 (1997) 365, [hep-th/9610238].

[73] C. Grojean, F. Quevedo, G. Tasinato and I. Zavala, JHEP 0108 (2001) 005, [hep-th/0106120].

[74] C. M. Hull, JHEP 9807 (1998) 021, [hep-th/9806146].

[75] N. S. Deger and A. Kaya, JHEP 0207 (2002) 038, [hep-th/0206057].

[76] N. Ohta, Phys. Lett. B 558 (2003) 213 [hep-th/0301095].

[77] N. Ohta, Phys. Lett. B 559 (2003) 270, [hep-th/0302140].

[78] N. Ohta, Phys. Lett. B 403 (1997) 218, [hep-th/9702164].

[79] R. Kallosh, Phys. Lett. B 282 (1992) 80, [hep-th/9201029].

[80] S. Ferrara, R. Kallosh, A. Strominger,
Phys. Rev. D 52 (1995) 5412, [hep-th/9508072].

[81] M. Cvetic, and A. A. Tseytlin, Phys. Lett. B 366 (1996) 95, [hep-th/9510097].

[82] K. Behrndt, G. L. Cardoso, B. de Wit, R. Kallosh, D. Lüst, and T. Mohaupt,
Nucl. Phys. B 488 (1997) 236, [hep-th/9601015].

[83] K. P. Tod, Class. Quant. Grav. 12 (1995) 1801.

[84] W.A. Sabra, Mod. Phys. Lett. A 13 (1997) 239, [hep-th/9708103].

[85] J.P. Gauntlett, R.C. Myers, and P.K. Townsend,
Class. Quantum Grav. 16 (1999) 1, hep-th/9809065.

[86] J.P. Gauntlett, J.B. Gutowski, C.M. Hull, S. Pakis and S. Reall,
Class. Quant. Grav. 20 (2003) 4587, [hep-th/0209114].

[87] J.P. Gauntlett, Fortsch. Phys. 53 (2005) 468, [hep-th/0501229].

[88] J.P. Gauntlett and S. Pakis,
Commun. Math. Phys. 247 (2004) 421, [hep-th/0212008].

[89] J.B. Gutowski, D. Martelli and H. S. Reall,
Class. Quant. Grav. 20 (2003) 5049, [hep-th/0306235].

[90] J.C. Breckenridge, R.C. Myers, A.W. Peet and C. Vafa,
Phys. Lett. B 391 (1993) 93, [hep-th/9602065].

[91] H.W. Brinkmann, Proc. Natl. Acad. Sci. U.S. 9 (1923) 1.

[92] H. Elvang, R. Emparan, D. Mateos, H.S. Reall,
Phys. Rev. Lett. 93 (2004) 211302, [hep-th/0407065].

[93] J.B. Gutowski, H.S. Reall, JHEP 0402 (2004) 006, [hep-th/0401042].

[94] J.P. Gauntlett and J.B. Gutowski, Phys. Rev. D71 (2005) 045002, [hep-th/048122].

[95] J. Maldacena and C. Nunez,
Int. J. Mod. Phys. A 16 (2001) 822-855, [hep-th/0007018].

[96] K. P. Tod, Phys. Lett. B 121 (1983) 241.

[97] Y. Nakayama, K. L. Panigrahi, S. Rey and H. Takayanagi, JHEP 0501 (2005) 052, [hep-th/0412038].

[98] H. Yavartanoo, Int. J. Mod. Phys. A 20 (2005) 7633, [hep-th/0407079].

[99] A. Sen, JHEP 0204 (2002) 048, [hep-th/0203211].

[100] I. Bena and N.P. Warner, Adv. Theor. Math. Phys. 9 (2005) 667, [hep-th/0408106].

[101] H. Elvang, R.Emparan, D. Mateos, H.S. Reall, Phys. Rev. D 71 (2005) 024033, [hep-th/0408120].

[102] I. Bena and P. Kraus, JHEP 0412 (2004) 070, [hep-th/0408186].

[103] V. A. Belinski and V. E. Zakharov, Sov. Phys. JETP 48 (1978) 985.

[104] V. A. Belinski and V. E. Zakharov, Sov. Phys. JETP 50 (1979) 1.

[105] T. Mishima and H. Iguchi, Phys. Rev. D 73 (2006) 044030, [hep-th/0504018].

[106] T. Koikawa, Prog. Theor. Phys. 114 (2005) 793, [hep-th/0501248].

[107] J. Boer, K. Hori, H. Ooguri and Y. Oz, Nucl. Phys. B 522 (1998) 20, [hep-th/9801060].

[108] T. Hertog and G. T. Horowitz, JHEP 0407 (2004) 073, [hep-th/0406134].

[109] L. Susskind, [hep-th/9309145].

[110] P. Claus, M. Derix, R. Kallosh, J. Kumar, P. K. Townsend, A. van Proeyen, Phys.Rev.Lett. 81 (1998) 4553-4556, [hep-th/9804177].

[111] G.W. Gibbons, Class. Quant. Grav. 20 (2003) S321.

[112] Gibbons and Hull, [hep-th/0111072].

[113] P. Frè, M. Trigiante and A. Van Proeyan, Class. Quant. Grav. 19 (2002) 4167, [hep-th/0205119].

[114] R. Emparan and J. Garriga, JHEP 0305 (2003) 028, [hep-th/0304124].